LECTURE 10

Continuous Bayes rule; Derived distributions

- **Readings:** Section 3.6; start Section 4.1

Review

$$
\begin{array}{rl}
p_X(x) & f_X(x) \\
p_{X,Y}(x,y) & f_{X,Y}(x,y) \\
p_{X|Y}(x \mid y) = \dfrac{p_{X,Y}(x,y)}{p_Y(y)} & f_{X|Y}(x \mid y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} \\
p_X(x) = \displaystyle\sum_y p_{X,Y}(x,y) & f_X(x) = \displaystyle\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy
\end{array}
$$

$$F_X(x) = \mathbf{P}(X \le x)$$

$$\mathbf{E}[X], \quad \text{var}(X)$$

The Bayes variations

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} = \frac{p_X(x)p_{Y|X}(y \mid x)}{p_Y(y)}$$

$$p_Y(y) = \sum_x p_X(x)p_{Y|X}(y \mid x)$$

Example:

- $X = 1, 0$: airplane present/not present
- $Y = 1, 0$: something did/did not register on radar

Continuous counterpart

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x)f_{Y|X}(y \mid x)}{f_Y(y)}$$

$$f_Y(y) = \int_x f_X(x)f_{Y|X}(y \mid x)\,dx$$

Example: X: some signal; "prior" $f_X(x)$
Y: noisy version of X
$f_{Y|X}(y \mid x)$: model of the noise

Discrete X, Continuous Y

$$p_{X|Y}(x \mid y) = \frac{p_X(x)f_{Y|X}(y \mid x)}{f_Y(y)}$$

$$f_Y(y) = \sum_x p_X(x)f_{Y|X}(y \mid x)$$

Example:

- X: a discrete signal; "prior" $p_X(x)$
- Y: noisy version of X
- $f_{Y|X}(y \mid x)$: continuous noise model

Continuous X, Discrete Y

$$f_{X|Y}(x \mid y) = \frac{f_X(x)p_{Y|X}(y \mid x)}{p_Y(y)}$$

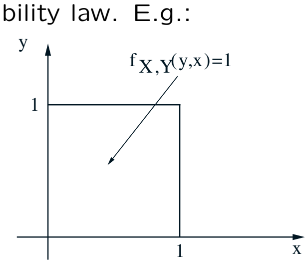
$$p_Y(y) = \int_x f_X(x)p_{Y|X}(y \mid x)\,dx$$

Example:

- X: a continuous signal; "prior" $f_X(x)$ (e.g., intensity of light beam);
- Y: discrete r.v. affected by X (e.g., photon count)
- $p_{Y|X}(y \mid x)$: model of the discrete r.v.

What is a derived distribution

- It is a PMF or PDF of a function of one or more random variables with known probability law. E.g.:



  – Obtaining the PDF for

$$g(X,Y) = Y/X$$

  involves deriving a distribution.
  Note: $g(X,Y)$ is a random variable

When not to find them

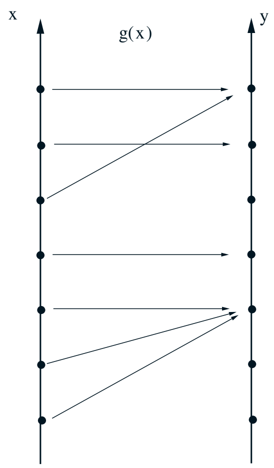
- Don't need PDF for $g(X,Y)$ if only want to compute expected value:

$$\mathbf{E}[g(X,Y)] = \int\int g(x,y)f_{X,Y}(x,y)\,dx\,dy$$

How to find them

- **Discrete case**
  - Obtain probability mass for each possible value of $Y = g(X)$

$$\begin{aligned} p_Y(y) &= \mathbf{P}(g(X) = y) \\ &= \sum_{x:\, g(x)=y} p_X(x) \end{aligned}$$



The continuous case

- **Two-step procedure:**
  - Get CDF of Y: $F_Y(y) = \mathbf{P}(Y \le y)$
  - Differentiate to get

$$f_Y(y) = \frac{dF_Y}{dy}(y)$$

Example

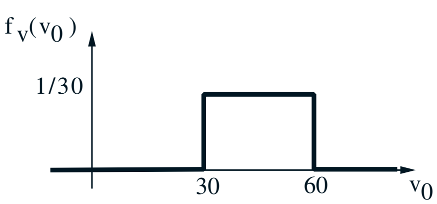
- X: uniform on [0,2]
- Find PDF of $Y = X^3$
- **Solution:**

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \le y) = \mathbf{P}(X^3 \le y) \\ &= \mathbf{P}(X \le y^{1/3}) = \frac{1}{2} y^{1/3} \end{aligned}$$

$$f_Y(y) = \frac{dF_Y}{dy}(y) = \frac{1}{6y^{2/3}}$$

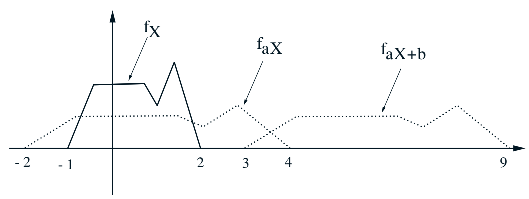
Example

- Joan is driving from Boston to New York. Her speed is uniformly distributed between 30 and 60 mph. What is the distribution of the duration of the trip?
- Let $T(V) = \dfrac{200}{V}$.
- Find $f_T(t)$



The pdf of Y=aX+b

$Y = 2X + 5$:



$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

- Use this to check that if X is normal, then $Y = aX + b$ is also normal.