Name: ______________________________

DEPARTMENT OF EECS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.02: Digital Communication Systems, Fall 2012

Quiz II

November 13, 2012

"×" your section	Section	Time	Recitation Instructor	TA
☐	1	10-11	Victor Zue	Ruben Madrigal
☐	2	11-12	Victor Zue	Cassandra Xia
☐	3	12-1	Jacob White	Kyu Seob Kim
☐	4	1-2	Yury Polyanskiy	Shao-Lun Huang
☐	5	2-3	Yury Polyanskiy	Rui Hu
☐	6	3-4	Jacob White	Eduardo Sverdlin Lisker

Please read and follow these instructions:

0. Please write your name in the space above and × your section.
1. Two two-sided "crib sheets" and a calculator are allowed. No other aids.
2. There are **27 questions** (mostly short!) in VI sections, and **17 pages** in this quiz booklet.
3. Your total allotted time is **120 minutes**.
4. **Please write legibly. Explain your answers, not just when we explicitly ask you to!** If you find a question ambiguous, write down your assumptions. Show your work for partial credit.
5. Use the empty sides of this booklet if you need scratch space. *If you use the blank sides for answers, make sure to say so!*

PLEASE NOTE: SOME STUDENTS WILL TAKE A MAKE-UP QUIZ LATER THAN YOU. PLEASE DON'T DISCUSS THIS QUIZ WITH ANYONE IN THE CLASS, UNLESS YOU'RE SURE THEY HAVE TAKEN IT WITH YOU TODAY.

Do not write in the boxes below

1-6(*/19)	7-10 (*/19)	11-17 (*/25)	18-21 (*/13)	22-24 (*/12)	25-27 (*/12)	Total (*/100)

I Bring in The Noise

A particular digital communication scheme sends signals over a channel that behaves essentially as a linear, time-invariant (LTI) system at baseband. The binary source at the transmitter generates the symbols "0" and "1" with equal probability. The characteristics of the channel and the timing at the receiver are such that the receiver is able to obtain $M \geq 1$ good samples in each bit slot. The good samples in a bit slot corresponding to a "0" are all of the form

$$y[n] = V_0 + w[n]$$

while in a bit slot corresponding to a "1" they are all of the form

$$y[n] = V_1 + w[n] \ .$$

Here $w[n]$ denotes a noise term that is a Gaussian random variable of mean 0 and variance σ^2, and is independent across samples, i.e., the samples at the receiver are perturbed by additive white Gaussian noise.

For on-off signaling, $V_0 = 0$ and $V_1 = V$. The linearity of the channel then ensures that for bipolar signaling $V_0 = -V$ and $V_1 = V$.

The receiver decides on whether a "0" or "1" was sent by comparing the *average* value of the M samples in any bit slot with a threshold voltage V_{th}. If the average is below V_{th}, the receiver decides a "0" was sent; if the average is above V_{th}, the receiver decides a "1" was sent.

Suppose we are using **bipolar** signaling and $M = 1$, i.e., only a single sample is taken in each bit slot. Then you've seen that the probability of the receiver making an error in deciding whether a "0" or "1" was sent in any particular bit slot, i.e., the bit error rate (BER), is minimized by choosing $V_{th} = 0$, with the corresponding BER being given by

$$\mathrm{BER}_{\mathrm{bipol}} = 0.5\,\mathrm{erfc}\Big(\frac{V}{\sigma\sqrt{2}}\Big) \ .$$

1. [2 points]: If V is doubled from its original value, by what maximum factor can the noise **variance** σ^2 be increased without exceeding the original BER?

Noise **variance** can be increased by a factor of :

2. [4 points]: If V is kept the same as originally, but M is increased to 4, by what maximum factor can the noise **variance** σ^2 be increased without increasing the original BER? **Explain your answer**!

Noise **variance** can be increased by a factor of :

Suppose the transmitter now converts to **on-off** signaling, perhaps to save on the average power transmitted, and thereby extend its battery life. Some indication of the average power at the transmitter can be obtained by computing the average value that $(y[n])^2$ would have across all the received good samples in the absence of noise. By this measure it follows that, for a given V, on-off signaling uses *half* the average power at the transmitter as bipolar signaling, when "0" and "1" are equally likely.

You can assume the receiver is properly converted for optimal (i.e, minimum BER) detection of on-off signaling by now using a threshold $V_{th} = V/2$.

3. [4 points]: For $M = 1$, what is the corresponding BER now? (You need not derive your answer in detail; it will suffice to **explain what modifications you need to make** to the earlier $\text{BER}_{\text{bipol}}$ expression, **and why**.)

$\text{BER}_{\text{onoff}} =$

4. [4 points]: For a given M and noise variance, by what factor does V in on-off signaling have to be increased in order to get a BER that matches the case of bipolar signaling under the original V?

V must be increased by a factor of :

5. [2 points]: If V for on-off signaling is increased by the factor you determined in the preceding problem, by what factor does the average power at the transmitter increase?

Average power at the transmitter increases by :

6. [3 points]: Suppose (for either bipolar or on-off signaling) the probability of a "1" being sent becomes greater than the probability of a "0" being sent. Will V_{th} have to be increased, decreased, or left unchanged from its previous value for optimum performance, i.e., minimum BER? Circle the correct answer, and explain!

V_{th} will have to be **Increased** / **Decreased** / **Unchanged**

II Signals and Systems

Consider the signal $x[n]$ given by

$$x[n] = \cos(2\pi \frac{n}{3})$$

for all n. A segment of the signal, for n in the interval $[-3, 3]$, is shown below. (Note that $\cos(\frac{2\pi}{3}) = -0.5$.)

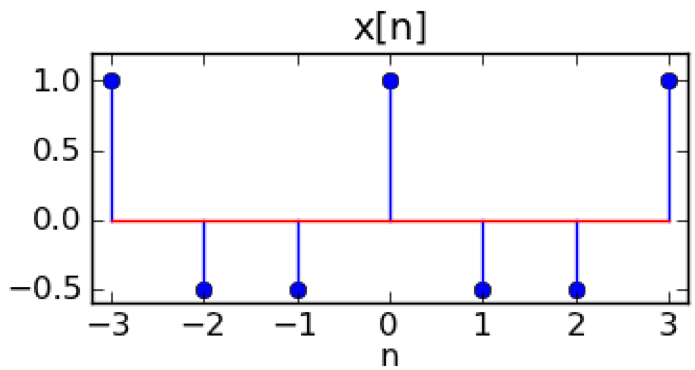

Figure 1: Input signal $x[n]$.

7. [3 points]: What is the angular frequency Ω_0, in radians/sample, of this sinusoidal signal $x[n]$? And what is its period?

$\Omega_0 =$ Period $=$

Suppose the above sinusoidal signal $x[n]$ is used as the input to an LTI system whose unit sample response is

$$h[n] = \delta[n] - \delta[n-1] + \delta[n-2] ,$$

where $\delta[n]$ as usual denotes the unit sample function. A plot of $h[n]$ for n in $[-3, 3]$ is shown on the next page.

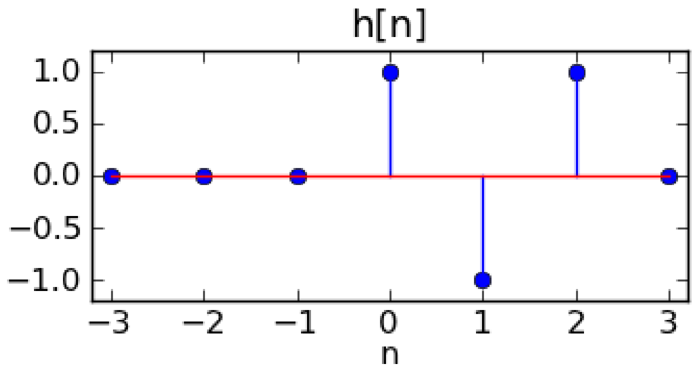

Figure 2: Unit sample response $h[n]$ of LTI system.

8. [6 points]: Determine the output $y[n]$ of the system for $n = 0, 1, 2$, showing your computations explicitly.

$y[0] =$, $y[1] =$, $y[2] =$

Now suppose instead that we use a sinusoidal input that has half the frequency of the preceding case. This input, which we'll denote by $x'[n]$, is given **for all** n by the expression

$$x'[n] = \cos(\pi \frac{n}{3})$$

and is plotted on the next page for n in $[-3, 3]$. (Note that $\cos(\frac{\pi}{3}) = 0.5$.)

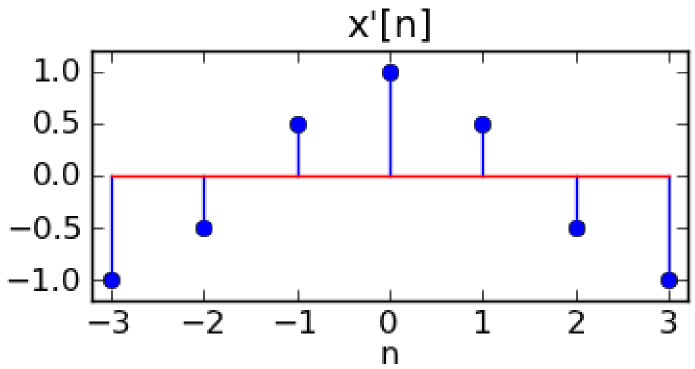

Figure 3: New input signal $x'[n]$.

This input is applied to the same system as before, namely the one with unit sample response

$$h[n] = \delta[n] - \delta[n-1] + \delta[n-2] \ .$$

9. [6 points]: Determine the output $y'[n]$ of the system for this new input, for $n = 0, 1, 2, 3, 4, 5$. You only need show your computations for the first three of these values.

$y'[0] =$, $y'[1] =$, $y'[2] =$

$y'[3] =$, $y'[4] =$, $y'[5] =$

Staying with the LTI system we have been considering, whose unit sample response is

$$h[\cdot] = \delta[n] - \delta[n-1] + \delta[n-2] \ ,$$

suppose some input signal $x_c[n]$ to this system produces an output of 0 for all time. In other words,

$$(h * x_c)[n] = 0$$

for all n. Consider [illegible] tem is a stable LTI system with some u [illegible] e one we have been considering.

Figure 4: Composite system.

10. [4 points]: What is the output $y_c[\cdot]$ of the composite system? **It is essential that you explain your reasoning here.**

III Frequently

We stick with the LTI system introduced in the previous section, namely the system with unit sample response $h[n] = \delta[n] - \delta[n-1] + \delta[n-2]$.

11. [3 points]: Write down an explicit expression for the frequency response

$$H(\Omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\Omega m}$$

of the particular LTI system specified above.

$H(\Omega) =$

12. [6 points]: Rewrite your answer from the previous part in the form

$$H(\Omega) = A(\Omega)e^{j\alpha(\Omega)} ,$$

where $\alpha(\Omega)$ and $A(\Omega)$ are **real** functions of Ω. [We don't insist that $A(\Omega)$ be nonnegative, so $A(\Omega)$ need not be the magnitude of $H(\Omega)$.]

$A(\Omega) =$, $\alpha(\Omega) =$

13. [4 points]: Evaluate $A(\Omega)$ and $\alpha(\Omega)$ explicitly for $\Omega = \frac{\pi}{3}$ and for $\Omega = \frac{2\pi}{3}$.

$A(\frac{\pi}{3}) =$, $\alpha(\frac{\pi}{3}) =$

$A(\frac{2\pi}{3}) =$, $\alpha(\frac{2\pi}{3}) =$

14. [4 points]: Provide a careful and fully labeled sketch of $A(\Omega)$ below, for Ω in the interval $[-\pi, \pi]$.

15. [3 points]: Suppose the input to the system is $x''[n] = \cos(2\pi \frac{n}{3} + \theta_0)$. Write down an explicit expression for the corresponding output $y''[n]$.

$y''[n] =$

16. [3 points]: Suppose the input to the system is now $x'''[n] = \cos(\pi \frac{n}{3} + \theta_0)$. Write down an explicit expression for the corresponding output $y'''[n]$.

$y'''[n] =$

17. [2 points]: For the special case of $\theta_0 = 0$ in the preceding two problems, show that you recover the results you obtained in the last section (Problems 8 and 9). If you don't, then you have some checking (or explaining!) to do.

IV Spectre in the Mirror

Given a real signal $x[\cdot]$ with DTFT

$$X(\Omega) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\Omega m} = |X(\Omega)|e^{j\angle X(\Omega)} \ ,$$

let $v[\cdot]$ denote the signal obtained by time-reversing $x[\cdot]$, so $v[n] = x[-n]$.

18. [4 points]: Let $V(\Omega)$ denote the DTFT of $v[\cdot]$. Which of the following equations correctly shows how to obtain $V(\Omega)$ from $X(\Omega)$? Note, incidentally, that $X(-\Omega) = X^*(\Omega)$, where the latter quantity is the complex conjugate of $X(\Omega)$. (Circle the correct answer, and explain your reasoning.)

- $V(\Omega) = X(\Omega)$
- $V(\Omega) = 1/X(\Omega)$
- $V(\Omega) = 1/X(-\Omega)$
- $V(\Omega) = X(-\Omega)$
- $V(\Omega) = -X(\Omega)$

Define the signal $r[\cdot]$ as the convolution of $x[\cdot]$ and $v[\cdot]$, so $r[n] = (x * v)[n]$. More explicitly,

$$r[n] = \sum_{m=-\infty}^{\infty} x[m]v[n-m] = \sum_{m=-\infty}^{\infty} x[m]x[m-n] \ . \tag{1}$$

The signal $r[\cdot]$ is called the **autocorrelation** function of $x[\cdot]$, and $r[n]$ is called the autocorrelation at *lag* n.

19. [4 points]: Denote the DTFT of $r[\cdot]$ by $R(\Omega)$. Write down an expression that shows how to obtain $R(\Omega)$ from $X(\Omega)$; the result of the previous problem is likely to be helpful. If you've done things correctly, you should find that $R(\Omega)$ is entirely determined by the **magnitude** of $X(\Omega)$. Please write your answer in terms of $|X(\Omega)|$.

In terms of $|X(\Omega)|$ we can write $R(\Omega) =$

20. [3 points]: State whether each of the following statements is **True** or **False**, with a very short explanation in each case.

- $R(\Omega)$ is real. **True / False**

- $R(\Omega)$ is an even function of Ω. **True / False**

- $R(\Omega)$ is nonnegative at all Ω. **True / False**

We know from the inverse DTFT that

$$r[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} R(\Omega)e^{j\Omega n} d\Omega \ ,$$

from which

$$r[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} R(\Omega)\, d\Omega \ .$$

21. [2 points]: In the latter equation, express $r[0]$ in terms of $x[\cdot]$ using Eq. (1), and express $R(\Omega)$ in terms of $|X(\Omega)|$ using the result of Problem 19. Write down the resulting equation; this equality is known as *Parseval's theorem* for a discrete-time signal.

Parseval's theorem:

(From what you've proved above, the following chain of reasoning is justified:

$$\begin{aligned}
|r[n]| &= \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} R(\Omega)e^{j\Omega n} d\Omega\Big| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |R(\Omega)e^{j\Omega n}| d\Omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |R(\Omega)|\, d\Omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} R(\Omega)\, d\Omega \\
&= r[0] \ .
\end{aligned}$$

This establishes that the autocorrelation function has its maximum magnitude when the lag is 0. You've used similar ideas for preamble detection.)

V Q to the Rescue

James Bond at the casino in Macau wants urgently to send to MI6 in London a signal $x[n]$ that is 0 for $n < 0$ and whose values for $n \geq 0$ contain vital information. To prevent the casino from snooping, Mr. Bond decides to instead transmit the DTFT $X(\Omega)$ of this signal (actually samples of the DTFT at points in the interval $[-\pi, \pi]$, computed using his Furiously Fast Transformer, but we shall assume these samples are so close together that he's effectively transmitting the DTFT itself). Unfortunately, what gets transmitted is just the **real part** of $X(\Omega)$. We'll lead you through how the Q Branch wizards at MI6 (some of whom are reported to have studied at MIT on the Cambridge-MIT Exchange) reconstructed $x[n]$ from just this information, and using the knowledge that $x[n] = 0$ for $n < 0$.

Any signal $x[n]$ can be written as the sum of two terms:

$$x[n] = \underbrace{\frac{x[n] + x[-n]}{2}}_{x_1[n]} + \underbrace{\frac{x[n] - x[-n]}{2}}_{x_2[n]} .$$

As indicated above, we shall call the first term $x_1[n]$ and the second term $x_2[n]$.

22. [4 points]: Either $x_1[n]$ or $x_2[n]$ is an even function of time, and the other is an odd function of time. Which is which? Circle the correct answer below and show your reasoning. (Recall that $f[n]$ is called an even function if $f[-n] = f[n]$; it is called an odd function if $f[-n] = -f[n]$.)

$x_1[n]$ is **Odd / Even** $x_2[n]$ is **Odd / Even**

The above decomposition is unique (we don't ask you to show that, though it's not hard) — i.e., there is no other decomposition of an arbitrary signal into the sum of an even part and an odd part.

23. [4 points]: Denote the DTFTs of the above signals $x_1[n]$ and $x_2[n]$ by $X_1(\Omega)$ and $X_2(\Omega)$ respectively. One of these DTFTs is purely real and the other one is purely imaginary. Which is which? Circle the correct answer and show your reasoning.

$X_1(\Omega)$ is **Real / Imaginary** $X_2(\Omega)$ is **Real / Imaginary**

It follows from the answers to the previous two problems that the real part of $X(\Omega)$ equals either $X_1(\Omega)$ or $X_2(\Omega)$ — you will know which from your answers above. Thus the Q Branch wizards can inverse transform what the casino sent them, to obtain either $x_1[n]$ or $x_2[n]$ — you will know which from your earlier answers.

24. [4 points]: How can you recover $x[n]$ from either $x_1[n]$ alone or $x_2[n]$ alone, using the fact that $x[n]$ is 0 for $n < 0$?

The general lesson here is that a one-sided signal (i.e., a signal that is identically zero for all n less than some n_0, or for all n greater than some n_0) is uniquely determined by just the real part (or equivalently, by just the imaginary part) of its DTFT.

VI Bring Me the Messenger

[Although this problem is phrased in continuous-time, it should be quite straightforward for you to do on the basis of the fundamentals you've learned in the setting of DT modulation/demodulation. If it helps you, you could think instead in terms of DT samples of the signals here, taken at some sampling rate f_s kHz, and then map each frequency f given here in kHz to a frequency of $\Omega = 2\pi f/f_s$ radians/sample. However, that's much more complication than is needed for this fairly simple and straightforward problem!]

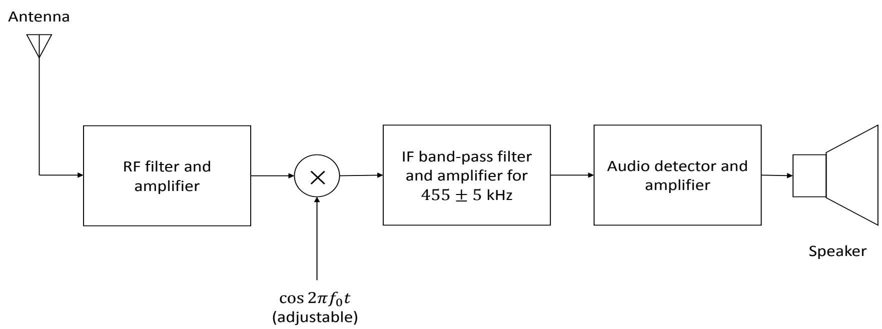

Figure 5: Structure of a typical radio receiver.

Practical radio receivers typically use several stages in their demodulation process, in order to manage various design tradeoffs. The baseband signal in AM is allowed to carry frequencies in the range of -5 kHz to 5 kHz, which is adequate for music and speech. To tune in an AM broadcast station operating at a carrier frequency of f_c kHz, we would like to select from the "radio frequency" (RF) signal picked up by the antenna only those frequencies in the ranges $f_c \pm 5$ kHz and $-f_c \pm 5$ kHz, as no other parts of the received signal are relevant. This would require building a very good bandpass filter for the antenna signal to go through, but the pass band of the filter would have to be shifted around in frequency as we tuned from one station to another — and this turns out to be tricky to do in hardware for a good bandpass filter.

Instead, what's done is to build an excellent bandpass filter whose passband is centered on a **fixed** pair of frequencies $\pm f_I$, referred to as the *intermediate frequency* or IF. For AM radio, the standard IF frequency is $f_I = 455$ kHz. This filter passes frequencies in the range $f_I \pm 5$ kHz and $-f_I \pm 5$ kHz, and effectively block out other frequencies. It can also include the amplification necessary to raise the signal level from the microvolts at the antenna to the higher levels needed to drive downstream electronics.

To move the desired radio station's signal into the passband of the IF filter/amplifier, we use the **heterodyning** idea that we've seen several times, i.e., multiply the signal at the antenna by a sinusoidal signal of appropriate frequency, f_o kHz. (In AM radio jargon, this multiplication is called mixing.) To tune in different stations, we just change the frequency of this sinusoidal signal, which is easy to do using a controllable oscillator. The common practice is to pick $f_o > f_c$; this is referred to as super-heterodyning.

25. [4 points]: Multiplying a carrier signal at f_c kHz with a receiver oscillator signal at f_o kHz produces a signal that is a sum of sinusoids. What are the frequencies of these sinusoids?

The sinusoids are at the following frequencies (in kHz):

26. [4 points]: What frequency f_o (in kHz, and with $f_o > f_c$) should the receiver's oscillator be set at in order to translate the signal received from an AM station operating at a carrier frequency of 580 kHz (which happens to be WTAG Worcester, in this area) into the passband of the IF filter/amplifier at 455 kHz? Explain!

27. [4 points]: With f_o set as in the preceding problem, there is actually another AM station, at a carrier frequency $f_c' > f_o$, that will also find its signal translated into the passband of the IF filter/amplifier. What is f_c'?

$f_c' =$

(This f_c' is referred to as the image frequency of f_c, and the way to prevent it interfering with the desired signal centered on f_c is to do some filtering at the antenna itself, in the RF stage. Also, the FCC is careful not to assign the carrier frequencies f_c and f_c' to nearby stations.)

The output from the IF filter/amplifier can now be demodulated in the standard ways, for instance using an averaging filter if the modulating signal is always nonnegative, otherwise using a further stage of mixing in which the signal is multiplied by a sinusoid at the IF frequency. But that's a story for another day.