6.011: Signals, Systems & Inference

Lec 3

Energy spectral density

Inner (dot) product of signals

$$< \mathbf{x}, \mathbf{v} >= \sum_k x[k]v[k] \quad \text{for real signals} \quad x[\cdot], v[\cdot]$$

$$\text{More generally,} \quad p[n] = \sum_k x[k]v[k-n] = x * \overleftarrow{v}[n]$$

$$\text{where} \quad \overleftarrow{v}[\ell] = v[-\ell]$$

Transform of reversed signal

$$\text{DTFT of } v[n] = V(e^{j\Omega}) = \sum_n v[n]e^{-j\Omega n}$$

$$\text{DTFT of } \overleftarrow{v}[n] = \sum_n v[-n]e^{-j\Omega n} = \sum_k v[k]e^{j\Omega k}$$

$$= V^*(e^{j\Omega}) = V(e^{-j\Omega})$$

Transform of inner product

$$\text{DTFT of } \overleftarrow{v}[n] = V(e^{-j\Omega}) \text{ so}$$

$$p[n] = \sum_k x[k]v[k-n] = x * \overleftarrow{v}[n]$$

$$\updownarrow$$

$$P(e^{j\Omega}) = X(e^{j\Omega})V(e^{-j\Omega})$$

Zero lag (n=0): Parseval, Rayleigh, Plancherel

$$p[0] = \sum_k x[k]v[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{j\Omega})d\Omega$$

so

$$\sum_k x[k]v[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})V(e^{-j\Omega})d\Omega$$

Energy

$$\text{Setting} \quad v[k] = x[k] \ ,$$

$$\mathcal{E}_x = \sum_k |x[k]|^2 = \frac{1}{2\pi} \int_{\pi}^{\pi} X(e^{j\Omega}) X(e^{-j\Omega}) d\Omega$$

$$= \frac{1}{2\pi} \int_{\pi}^{\pi} |X(e^{j\Omega})|^2 d\Omega$$

Energy spectral density (ESD)

$$\sum_k x[k]x[k \quad n] = \underbrace{\bar{R}_{xx}[n]}_{\textbf{deterministic autocorrelation}}$$

$$\updownarrow \quad \text{transform pair}$$

$$X(e^{j\Omega})^{2} = \underbrace{\bar{S}_{xx}(e^{j\Omega})}_{\textbf{energy spectral density}}$$

Cross (energy) spectral density

$$\bar{R}_{yx}[n] = \sum_k y[k]x[k-n] = y * \overleftarrow{x}[n]$$

$$\updownarrow$$

$$\bar{S}_{yx}(e^{j\Omega}) = Y(e^{j\Omega})X(e^{-j\Omega})$$

So if $y[n] = h * x[n]$ then

$$\bar{S}_{yx}(e^{j\Omega}) = H(e^{j\Omega})X(e^{j\Omega})X(e^{-j\Omega}) = H(e^{j\Omega})\bar{S}_{xx}(e^{j\Omega})$$

Similarly …

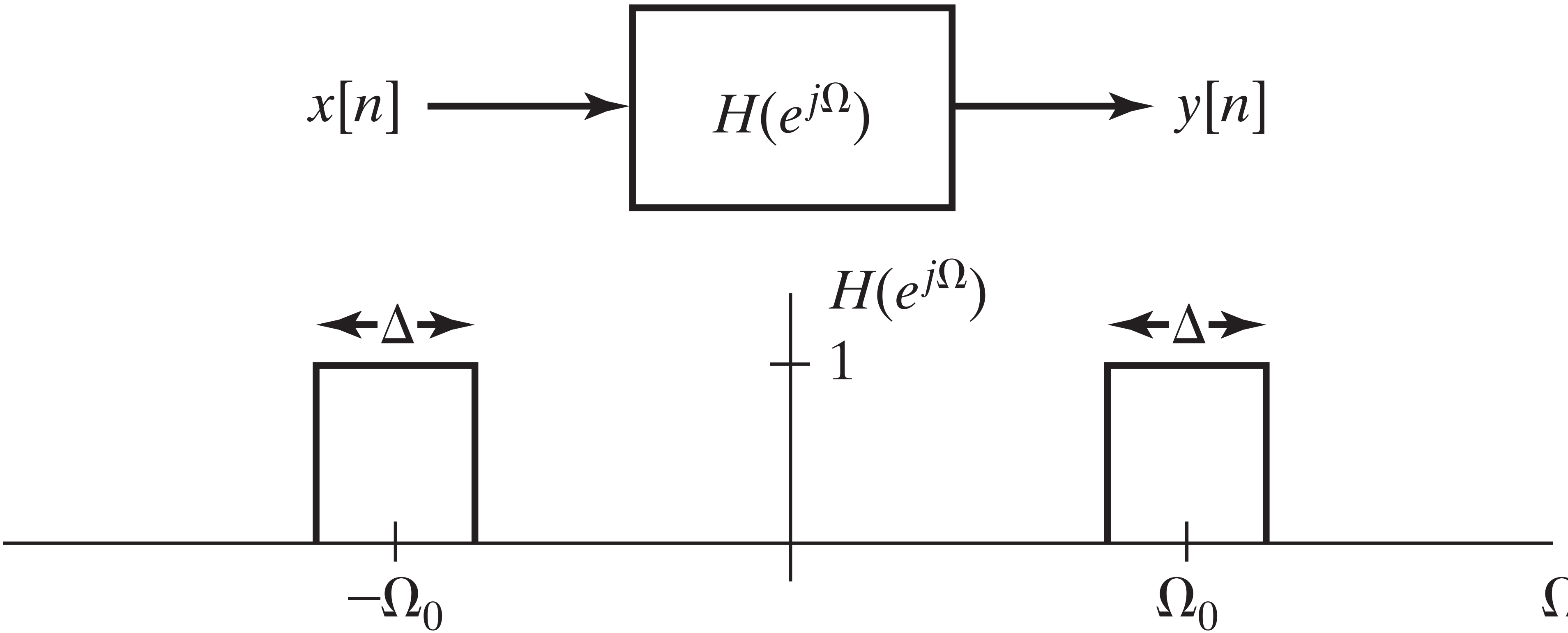
$$\ldots \text{ if } \; y[n] = h * x[n] \; \text{ then}$$

$$\bar{S}_{yy}(e^{j\Omega}) = Y(e^{j\Omega})Y(e^{-j\Omega})$$

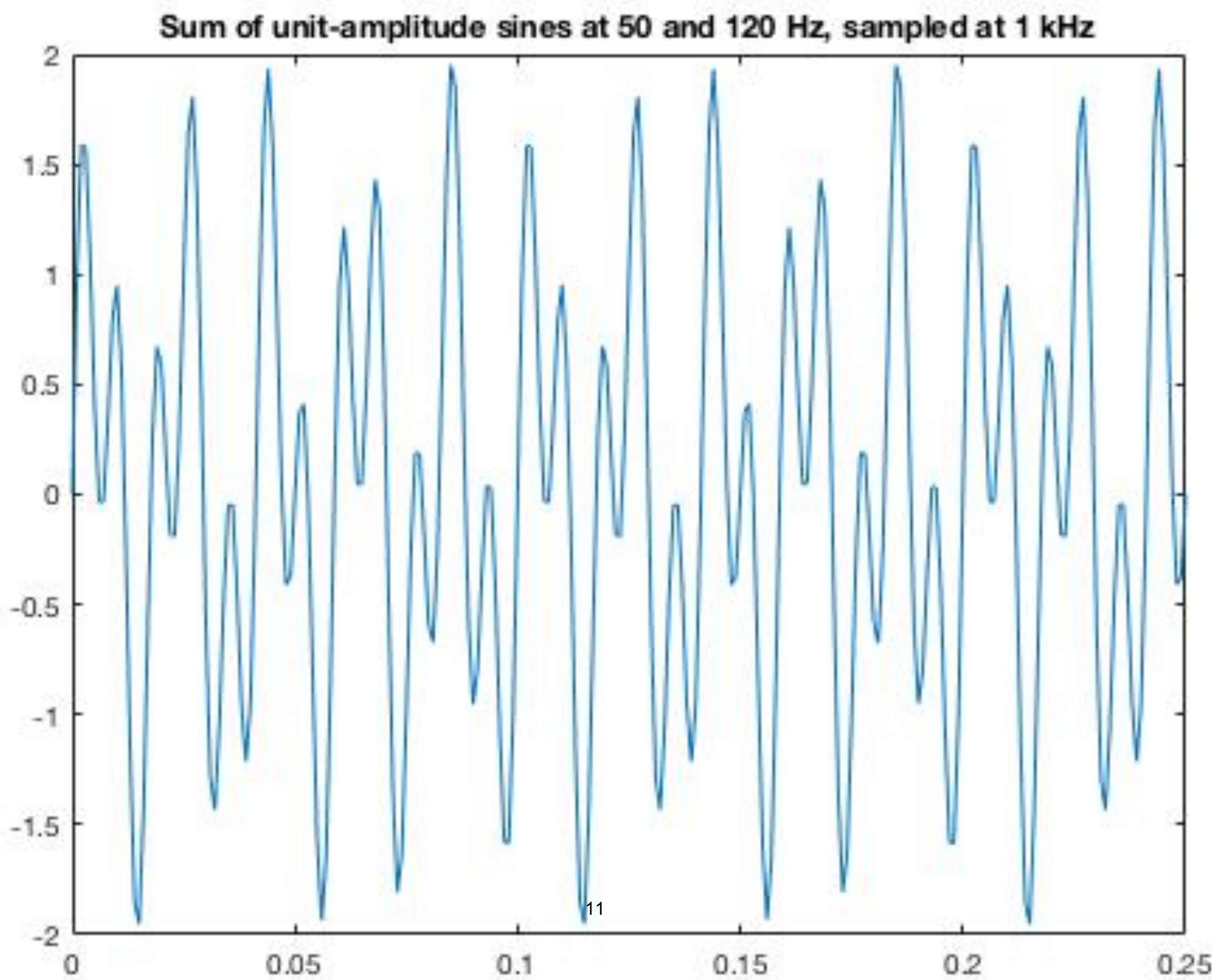
$$= H(e^{j\Omega})X(e^{j\Omega})X(e^{-j\Omega})H(e^{-j\Omega})$$

$$= \left|H(e^{j\Omega})\right|^2 \bar{S}_{xx}(e^{j\Omega})$$

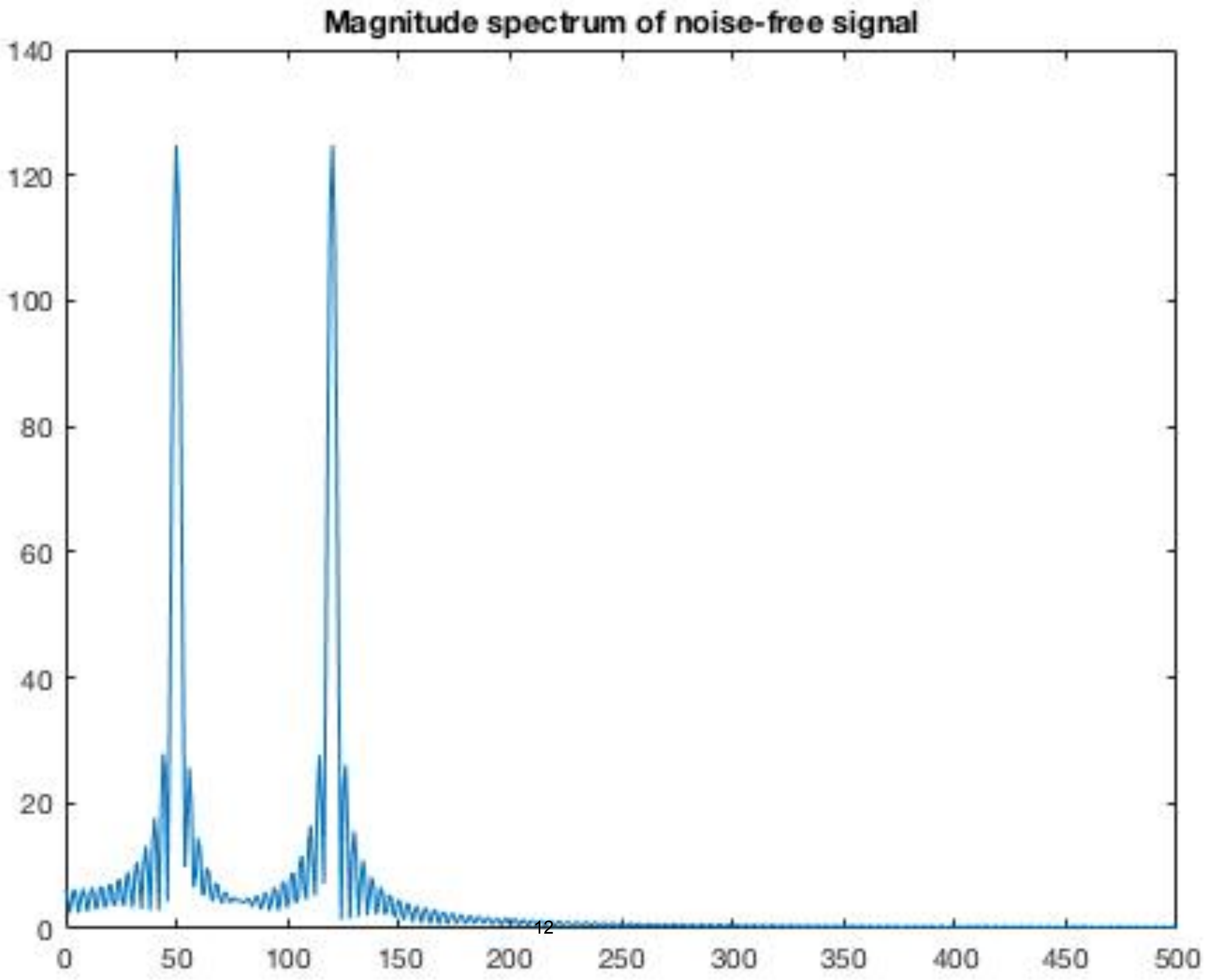
Energy of x[.] in a specified band



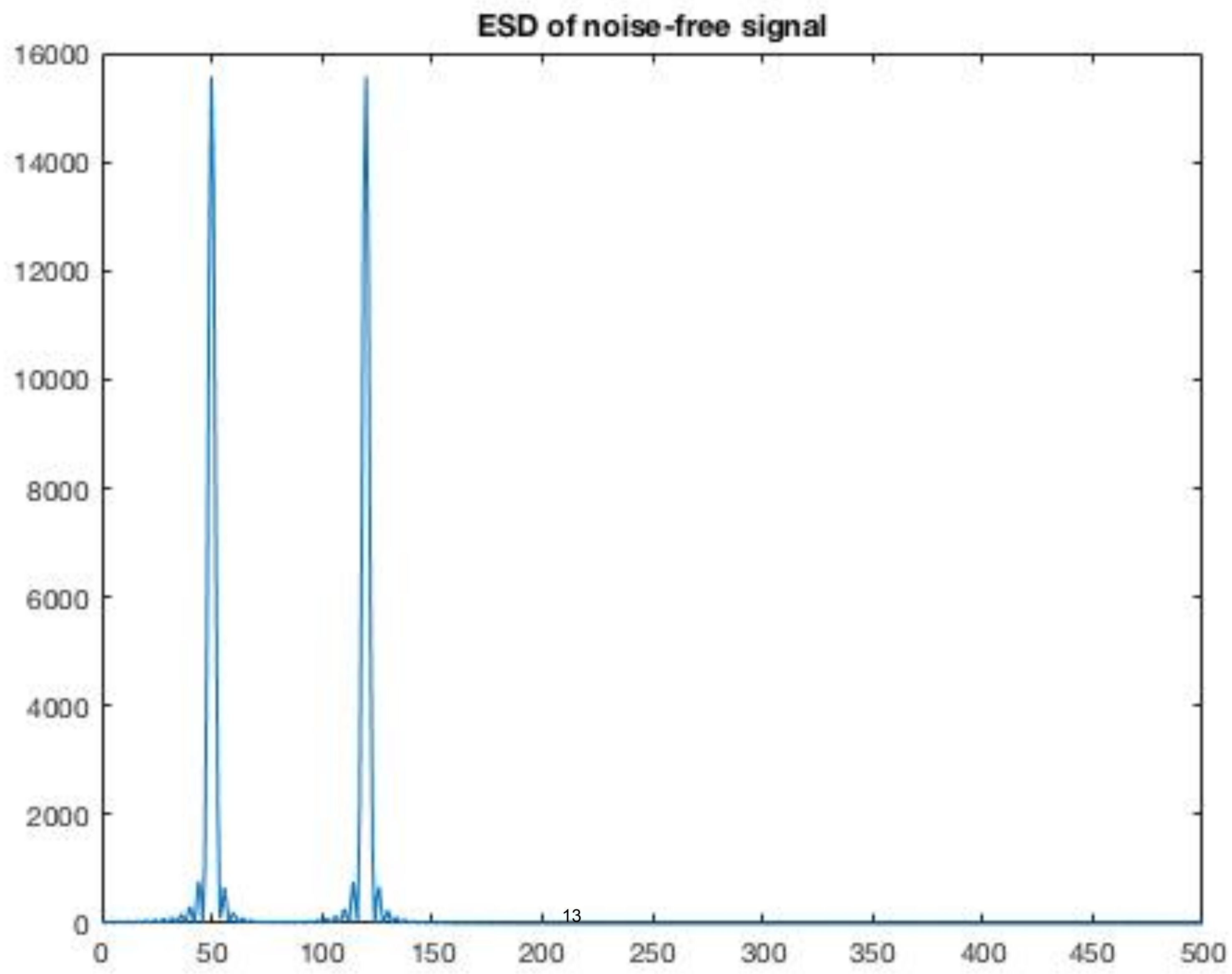
Noise-free signal



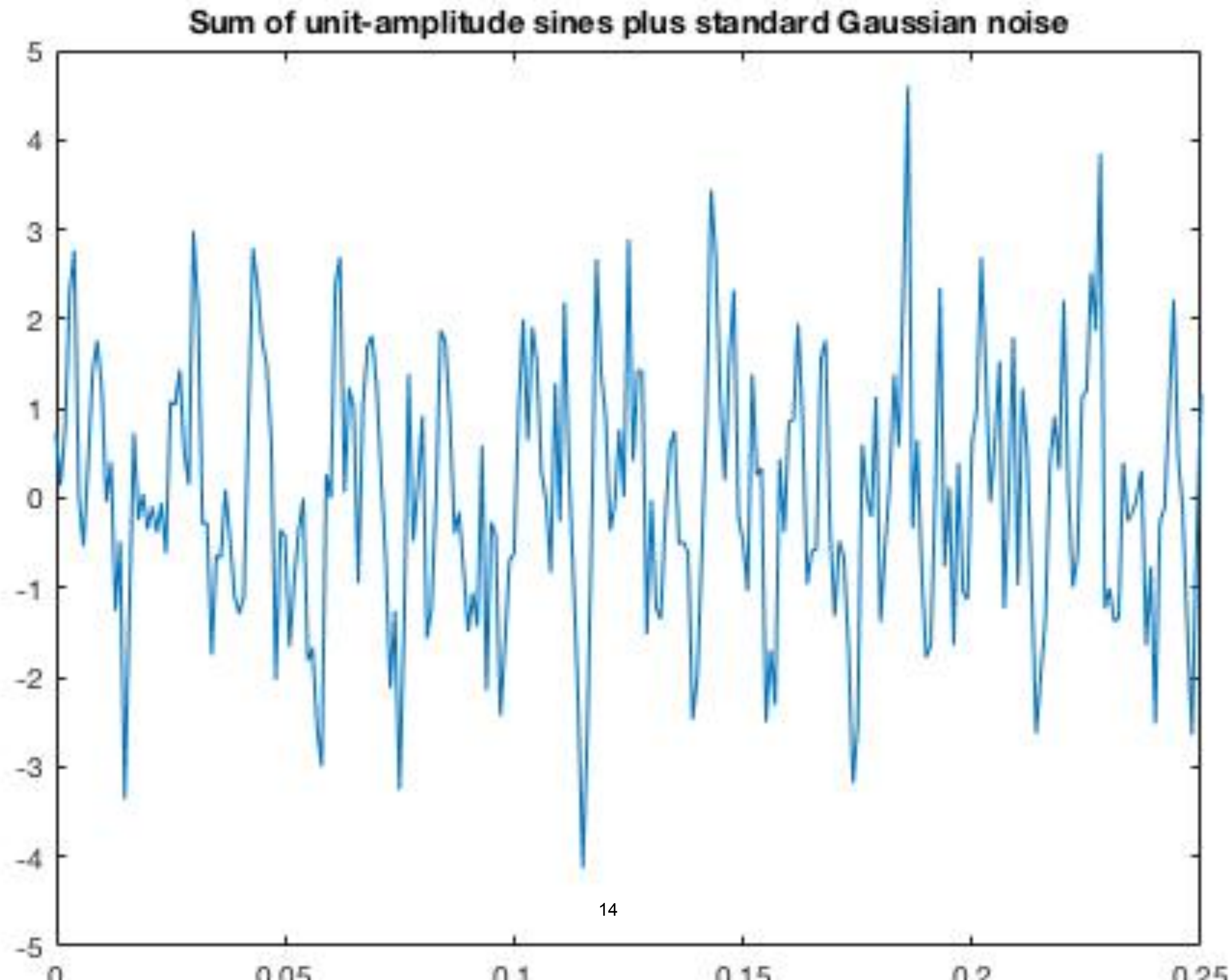
Magnitude spectrum of noise-free signal



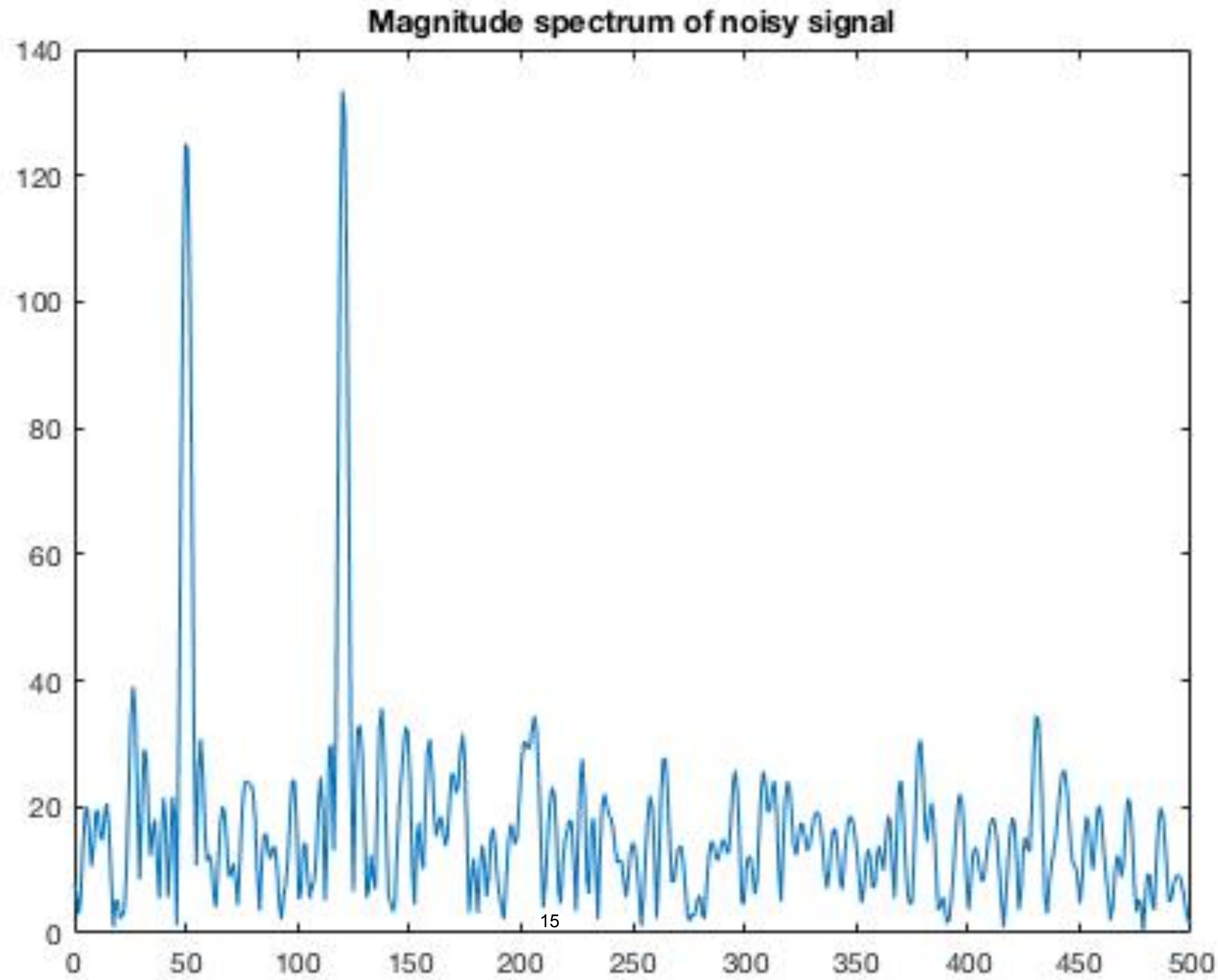
ESD of noise-free signal



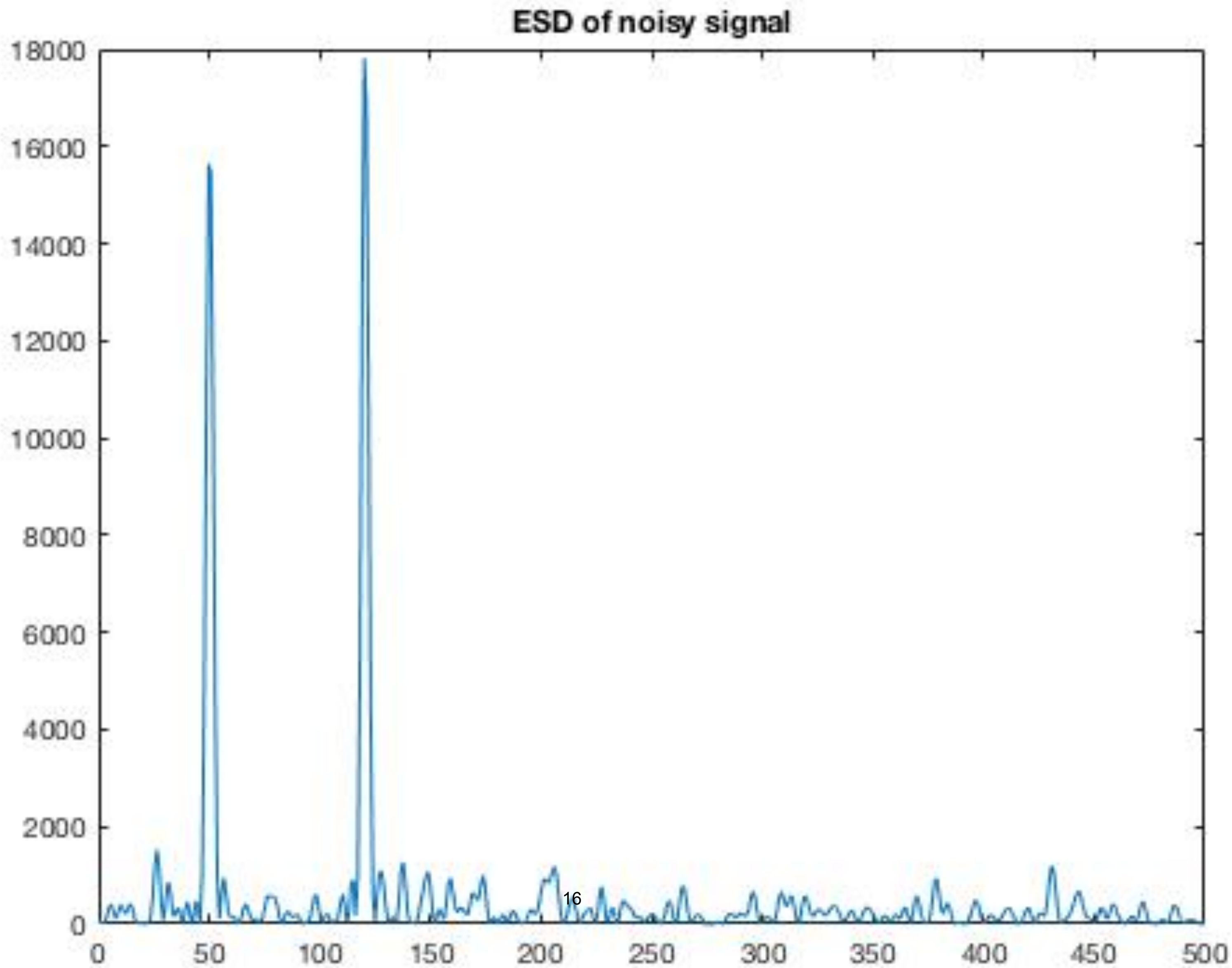
+ unit-intensity white Gaussian noise



Magnitude spectrum of noisy signal



ESD of noisy signal



Heart rate variability