Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Signals, Systems & Inference

Quiz 1

QUESTION & ANSWER BOOKLET

Your Full Name:	
Email :	

Please read this page carefully; it has important information!

This exam is closed book, but 2 sheets of notes (4 sides) are allowed. Calculators and other electronic aids will not be necessary and are not allowed. There are 3 problems with various subparts, weighted as shown, and for a total of 60 points. The points indicated on the following pages for the various subparts of the problems are our best guess for now, but may be modified slightly when we get to grading.

Check that this booklet has pages numbered up to 14. The booklet contains spaces for all relevant work and reasoning **BUT don't forget to read the actual questions! – reading only the short prompts in the answer spaces can lead you astray!** The work you want us to look at should be only in the indicated spaces. You can use scratch paper for rough work. Neat work and clear explanations count; show all relevant work and reasoning!

If you get stuck on something, try and move on!

Problem	Your Score
1 (18 points)	
2 (23 points)	
3 (19 points)	
Total (60 points)	

Problem 1 (18 points)

The figure below shows our standard configuration for DT processing of a CT signal $x_c(t)$ to produce a CT signal $y_c(t)$. The C/D box samples the CT waveform; the D/C box performs ideal bandlimited sinc interpolation. [To minimize the reading you have to do before getting to the questions on this, we've put a more detailed reminder of the specification of this system at the end of this problem, if you feel you need it.]
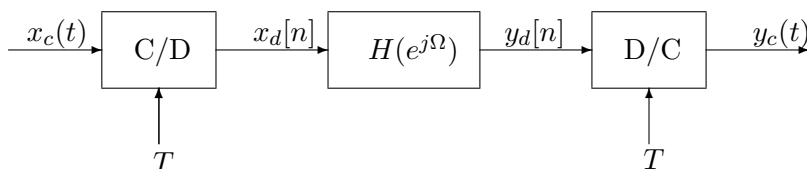
Assume throughout this problem that $x_c(t)$ is bandlimited to the frequency range $|\omega| < \pi/T$, and that the frequency response of the DT LTI filter is

$$H(e^{j\Omega}) = j\frac{\Omega}{T}, \quad |\Omega| < \pi .$$

As always, this frequency response repeats periodically with period 2π (so it is discontinuous at values of Ω that are odd multiples of $\pm\pi$, e.g., it is π/T just below $\Omega = \pi$ and it is $-\pi/T$ just above $\Omega = \pi$).

(a) (9 points) Denote the (real-valued) unit sample response of the DT filter by $h[n]$. Without computing $h[n]$ itself, answer the following and explain your reasoning (but don't get bogged down here — if you can't figure out some part, move on to the next!):

   (i) $\sum_n h[n] = ?$

   (ii) $\sum_n |h[n]|^2 = ?$

   (iii) What does $H(e^{j\Omega})$ being purely imaginary tell you about $h[n]$?

   (iv) $h[0] = ?$

   (v) Is $h[n]$ absolutely summable, i.e., is $\sum_n |h[n]|$ finite?

(b) (9 points) Let $X_c(j\omega)$ and $Y_c(j\omega)$ denote the transforms of $x_c(t)$ and $y_c(t)$ respectively.

(i) Determine the ratio $Y_c(j\omega)/X_c(j\omega)$ for $|\omega| < \pi/T$. (The following two subparts depend on your getting this right, so pay extra attention!)

(ii) Suppose the energy spectral density of $x_c(t)$ is $\overline{S}_{xx}(j\omega) = 3$ across the frequency range $|\omega| < \pi/T$ (and 0 outside, of course). Find an expression for the energy spectral density $\overline{S}_{yy}(j\omega)$ of $y_c(t)$ as a function of ω, for $-\infty < \omega < \infty$.

(iii) For the given system, $y_c(t)$ can be obtained from $x_c(t)$ by a simple mathematical operation. What operation?

Here's a more detailed description of the components of the system considered in this problem, if you need a reminder:

The C/D converter samples its input at integer multiples of T:

$$x_d[n] = x_c(nT) .$$

The DT processing here involves filtering by an LTI system with frequency response $H(e^{j\Omega})$. We assume an ideal D/C converter that performs band-limited sinc interpolation of its input samples with reconstruction interval T, so

$$y_c(t) = \sum_{n=-\infty}^{\infty} y_d[n] \frac{\sin(\pi(t - nT)/T)}{\pi(t - nT)/T} .$$

Space for rough work below. Begin your answers on next page.

1(a) (9 points) With $H(e^{j\Omega}) = j\frac{\Omega}{T}$, $|\Omega| < \pi$, determine the following:

(i) $\sum_n h[n] =$

Explanation:

(ii) $\sum_n |h[n]|^2 =$

Explanation:

(iii) What does $H(e^{j\Omega})$ being purely imaginary tell you about $h[n]$?

Answer:

(iv) $h[0] =$

Explanation:

(v) Is $h[n]$ absolutely summable, i.e., is $\sum_n |h[n]|$ finite?

Answer and explanation:

1(b) (9 points)

(i) For $|\omega| < \pi/T$,

$$\frac{Y_c(j\omega)}{X_c(j\omega)} =$$

Explanation:

(ii) Suppose $\overline{S}_{xx}(j\omega) = 3$ for $|\omega| < \pi/T$, and is 0 outside this range. For $-\infty < \omega < \infty$,

$$\overline{S}_{yy}(j\omega) =$$

Explanation:

1(b)(iii) What simple operation on $x_c(t)$ yields $y_c(t)$?

Answer and explanation:

Space below for additional rough work:

Problem 2 (23 points)

Part (e) of this problem, designing an observer, can be done independently of the earlier parts of this problem.

A particular mechanical system involves a single mass whose position $w(t)$ is governed by the differential equation

$$\frac{d^2}{dt^2}w(t) + \left(w^2(t) - 3\right)\frac{d}{dt}w(t) + 2w(t) = x^2(t) \,,$$

where $x(t)$ denotes the input to the system (but notice that this input is squared in the above equation) and t is time in seconds. Suppose the output of interest is

$$y(t) = \frac{d}{dt}w(t) \,.$$

(a) (5 points) Obtain a continuous-time 2nd-order state-space description of the above system, of the form

$$\frac{d}{dt}\mathbf{q}(t) = \mathbf{f}\Big(\mathbf{q}(t), x(t)\Big) \,, \qquad y(t) = g\Big(\mathbf{q}(t), x(t)\Big)$$

for an appropriately chosen state vector $\mathbf{q}(t)$. (Do this carefully, as parts (b)-(d) build on this!)

(b) (3 points) Suppose $x(t) = \bar{x}$, a constant, for all t. Determine the corresponding equilibrium values $\bar{\mathbf{q}}$ and $\bar{y}$ of respectively the state and output of your model in (a) in terms of $\bar{x}$.

(c) (6 points) Obtain a linearized state-space model of the form

$$\frac{d}{dt}\check{\mathbf{q}}(t) = \mathbf{A}\check{\mathbf{q}}(t) + \mathbf{b}\check{x}(t) \,, \qquad \check{y}(t) = \mathbf{c}^T\check{\mathbf{q}}(t) + \mathtt{d}\check{x}(t)$$

that approximately describes small perturbations from the equilibrium values you computed in (b), where we have used the notation $\check{\mathbf{q}}(t) = \mathbf{q}(t) - \bar{\mathbf{q}}$, and similarly for $\check{x}(t)$ and $\check{y}(t)$. Specify the coefficient matrices $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}^T$, $\mathtt{d}$ completely.

(d) (3 points) If $\bar{x} = 0$ in (b), what are the eigenvalues governing your linearized model in (c), and is the system asymptotically stable?

(e) (6 points) Depending on how you picked $\mathbf{q}(t)$ in (a), you may or may not find that the corresponding linearized model in (c) for a particular $\bar{x}$ has the following coefficient matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -61 \end{bmatrix} \,, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \,, \quad \mathbf{c}^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \,, \quad \mathtt{d} = 0 \,.$$

[If there is no choice of $\overline{x}$ that will result in the above matrices for your model in (c), it may just be that you've picked $\mathbf{q}(t)$ differently than we have, which is fine — but it's still worth doing a quick check of your work. In any case, use the particular matrices above for this part.]

Design an observer for the linearized model, using the measured $\check{x}(t)$ and $\check{y}(t)$ to produce an estimate $\widehat{\mathbf{q}}(t)$ of the state vector $\check{\mathbf{q}}(t)$ of the linearized model. Specifically, determine what choice of observer gains will result in the error dynamics being governed by eigenvalues at -10 and -20. If we consider a decaying exponential to have essentially settled after 3 time constants (it actually is just under 5% of its initial value then), around how long does it take for the observer error to settle?

Begin answers here:

2(a) (5 points) Given $\frac{d^2}{dt^2}w(t) + \left(w^2(t) - 3\right)\frac{d}{dt}w(t) + 2w(t) = x^2(t)$ (notice that the input $x(t)$ is squared!) and $y(t) = \frac{d}{dt}w(t)$, a possible choice state variables is:

$q_1(t) = \qquad\qquad\qquad q_2(t) =$

Resulting state evolution equation and output equation
$\frac{d}{dt}\mathbf{q}(t) = \mathbf{f}\Big(\mathbf{q}(t), x(t)\Big), \quad y(t) = g\Big(\mathbf{q}(t), x(t)\Big)$ are:

2(b) (3 points) The equilibrium values $\overline{\mathbf{q}}$ and $\overline{y}$ corresponding to $x(t) = \overline{x}$:

$\overline{q}_1 =$ $\qquad\qquad\qquad$ $\overline{q}_2 =$ $\qquad\qquad\qquad$ $\overline{y} =$

Calculations:

2(c)(6 points) Specify the coefficient matrices $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}^T$, d of the linearized model obtained at the equilibrium you computed in (b):

$\mathbf{A} =$ $\qquad\qquad\qquad$, $\quad \mathbf{b} =$ $\qquad$, $\quad \mathbf{c}^T =$ $\qquad\qquad$, $\quad \mathsf{d} =$

2(d) (3 points) If $\bar{x} = 0$ in (b), the eigenvalues of the linearized model in (c) are

$$\lambda_1 = \qquad , \quad \lambda_2 =$$

and the system **IS** / **ISN'T** asymptotically stable.

Calculations and explanation:

2(e) (6 points) Design an observer for a CT LTI system defined by the following coefficient matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -61 \end{bmatrix} , \quad \mathbf{b} = \begin{bmatrix} 0 \\ 8 \end{bmatrix} , \quad \mathbf{c}^T = \begin{bmatrix} 0 & 1 \end{bmatrix} , \quad \mathtt{d} = 0 ,$$

placing the error decay eigenvalues at -10 and -20. Also determine the approximate settling time of the error.

Observer gains are $\ell_1 = \qquad , \qquad \ell_2 =$

Approximate settling time is

Problem 3 (19 points)

Parts (a), (b) and (c) are completely independent of each other.

(a) (6 points) Suppose we are given a 3rd-order reachable LTI CT system that is initially in the zero state, i.e., $\mathbf{q}(0) = \mathbf{0}$. Is it possible to select a nonzero input $x(t)$ such that the motion of the state away from the origin $\mathbf{0}$ is confined entirely to motion along the direction of the first eigenvector, i.e., is it possible to get $\mathbf{q}(t) = r_1(t)\mathbf{v}_1$ for all t and some $r_1(t)$? If so, explain how; otherwise explain why not. (Don't try and get by with just intuition on this! — start with the equations that govern the motion of the state.)

(b) (7 points) Consider a reachable and observable L-th order DT LTI state-space system of the form

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] \,, \qquad y[n] = \mathbf{c}^T\mathbf{q}[n] \,,$$

where $\mathbf{A}$ has distinct eigenvalues $\{\lambda_i\}_1^L$ and associated eigenvectors $\{\mathbf{v}_i\}_1^L$. We know the transfer function of such a system has the form

$$H(z) = \frac{p_1}{z - \lambda_1} + \frac{p_2}{z - \lambda_2} + \cdots + \frac{p_L}{z - \lambda_L}$$

for some constants $\{p_i\}_1^L$. The corresponding unit sample response with zero initial conditions, i.e., the output $y[n]$ when $x[n] = \delta[n]$ and $\mathbf{q}[0] = \mathbf{0}$, is

$$h[n] = \left(p_1\lambda_1^{n-1} + p_2\lambda_2^{n-1} + \cdots + p_L\lambda_L^{n-1}\right)u[n-1] \,,$$

where $u[n]$ is the unit step function.

(i) What does the reachability and observability of the system tell you about the coefficients p_i?

(ii) Suppose the input is again the unit sample function, i.e., $x[n] = \delta[n]$, but that the initial state is picked to be some nonzero value $\mathbf{q}[0]$. The output at time 0 then becomes $y[0] = \mathbf{c}^T\mathbf{q}[0]$. What choice of $\mathbf{q}[0]$ will ensure that the output for $n \geq 1$ is

$$y[n] = p_2\lambda_2^{n-1} + \cdots + p_L\lambda_L^{n-1} \,,$$

i.e., the same as the zero-state unit impulse response, but with the first mode absent? Express and explain your answer in terms of the symbols introduced above.

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(c) (6 points) Suppose

$$e^{\mathbf{A}t} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

Determine

$$\frac{d}{dt}e^{\mathbf{A}t} \quad \text{and} \quad \mathbf{A}.$$

(As a check, $\mathbf{A}$ will be quite simple, with each entry being 0, 1, or -1.)

3(a) (6 points) 3rd-order reachable LTI CT system with $\mathbf{q}(0) = \mathbf{0}$. Is it possible to get $\mathbf{q}(t) = r_1(t)\mathbf{v}_1$ for all t and some $r_1(t)$? If so, explain how; otherwise explain why not. (Don't try and get by with just intuition on this! — start with the equations that govern the motion of the state.)

3(b)(7 points) Reachable and observable LTI DT state-space system
with unit sample response $h[n] = \left(p_1 \lambda_1^{n-1} + p_2 \lambda_2^{n-1} + \cdots + p_L \lambda_L^{n-1} \right) u[n-1]$.

(i) What does the reachability and observability of the system tell you
about the coefficients p_i?

Answer and explanation:

(ii) Suppose $x[n] = \delta[n]$ again, but that the initial state is picked to be
some nonzero value $\mathbf{q}[0]$. What choice of $\mathbf{q}[0]$ will ensure that the
output for $n \geq 1$ is $y[n] = p_2 \lambda_2^{n-1} + \cdots + p_L \lambda_L^{n-1}$?

Calculation and answer:

$\mathbf{q}[0] =$

3(c) **(6 points)** Suppose $e^{\mathbf{A}t} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. Determine $\frac{d}{dt}e^{\mathbf{A}t}$ and $\mathbf{A}$. (As a check, $\mathbf{A}$ will be quite simple, with each entry being 0, 1, or -1.)

$\frac{d}{dt}e^{\mathbf{A}t} =$ $\mathbf{A} =$