Stored Energy and Forces on Solenoids

(derived with the Energy Method)

Outline

Lorentz Force on a Coil
Energy Method for Calculating Force
Examples

What Sets the Limit ?

Pressure Under Water

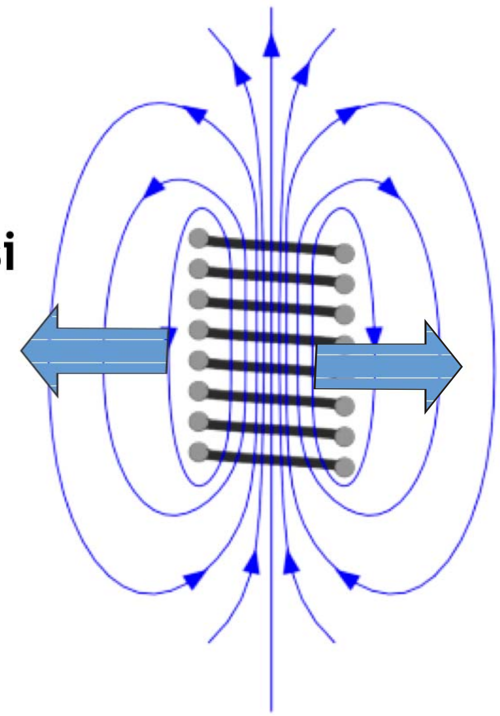
1000 m	**Submarine**	**1000 psi**
4000 m	**Ocean Floor Submersible**	**6000 psi**

80T Pulsed Magnet **200,000 psi**

(1.3GPascals, 130 kg/mm²)

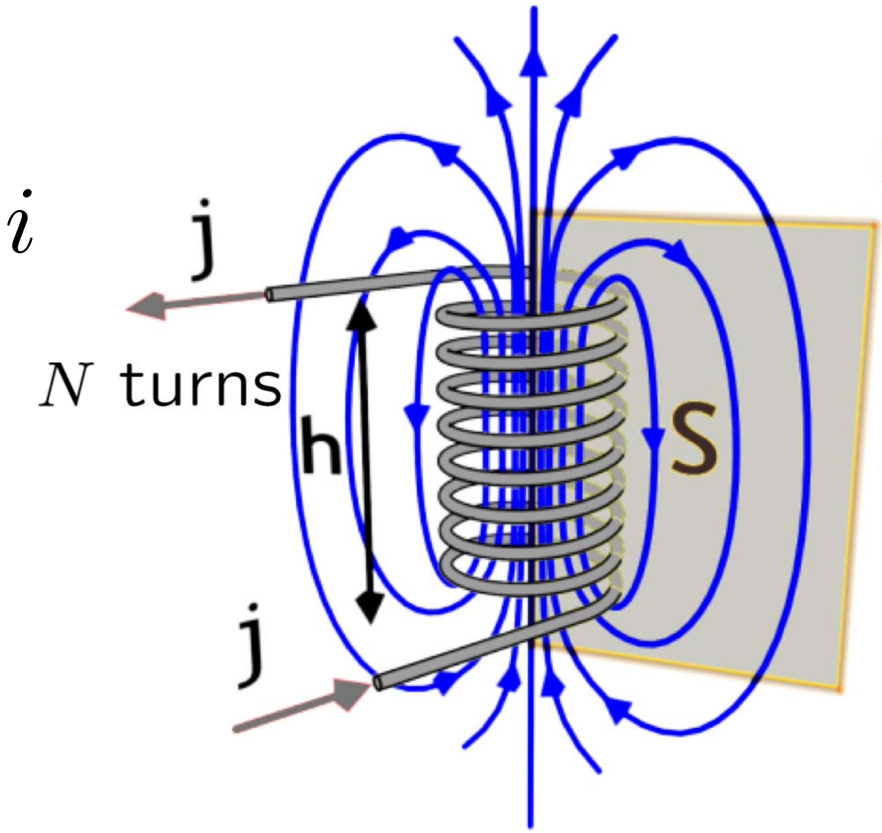
...exceeds the strength of most materials...

...within a factor of three of theoretical ultimate tensile strength...



Strong
Electromagnets
Generate
HUGE Forces

Review of Solenoids



For a sufficiently long solenoid...

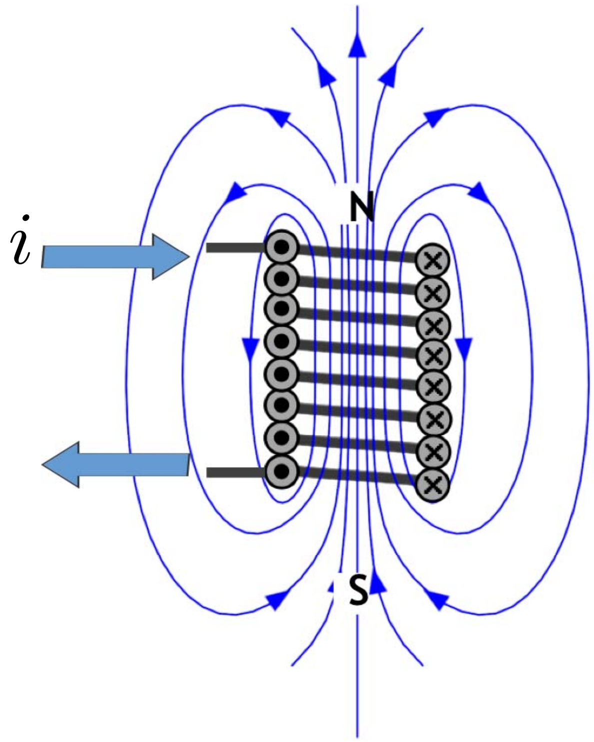
$$H_{inside} \approx \frac{Ni}{h} = ni$$

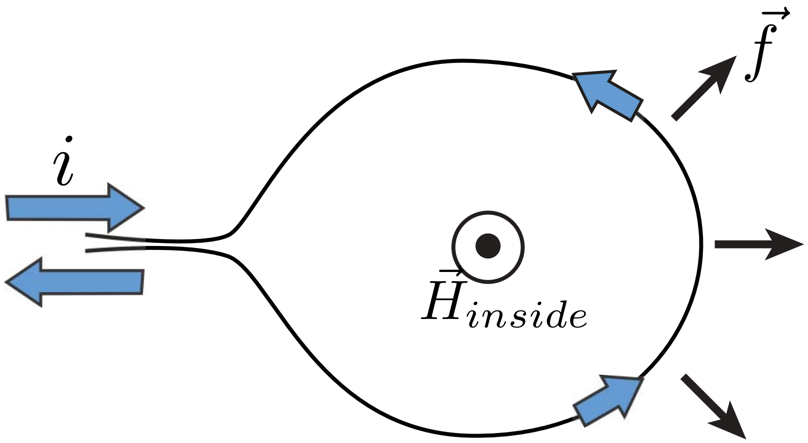
$$v = \frac{d\Phi}{dt} = \frac{d(Li)}{dt} = L\frac{di}{dt}$$

$$\Rightarrow L = \frac{\mu_o N^2 A}{h}$$

Qualitative Analysis of the Force

$$\vec{f} = q(\vec{E} + \vec{v} \times \vec{B})$$



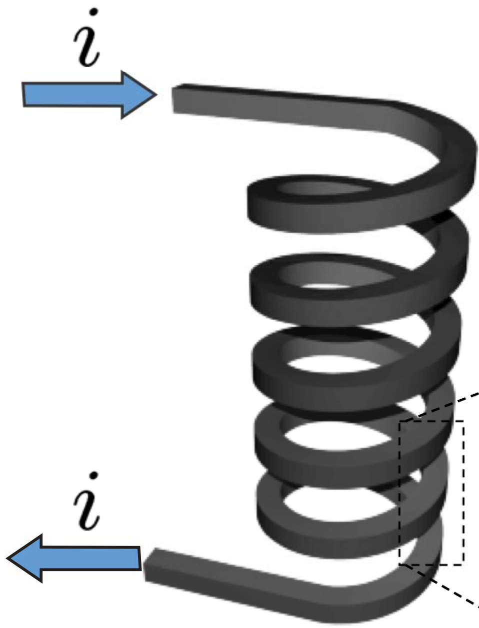


$$\vec{f} = q(\vec{v} \times \vec{B}) \Rightarrow \vec{F} = (\vec{J} \times \vec{B})$$

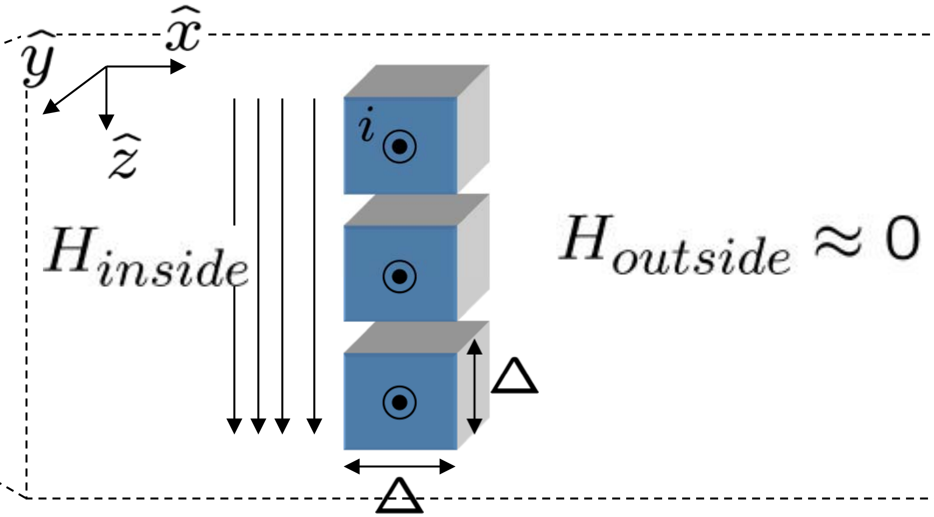
force density [N/m^3]

...need to know the field **inside** the wire

The H-fields Inside the Solenoid Wire



Lets assume that in cross-section each of the wires that forms the solenoid is a little square, with dimension Δ on a side:



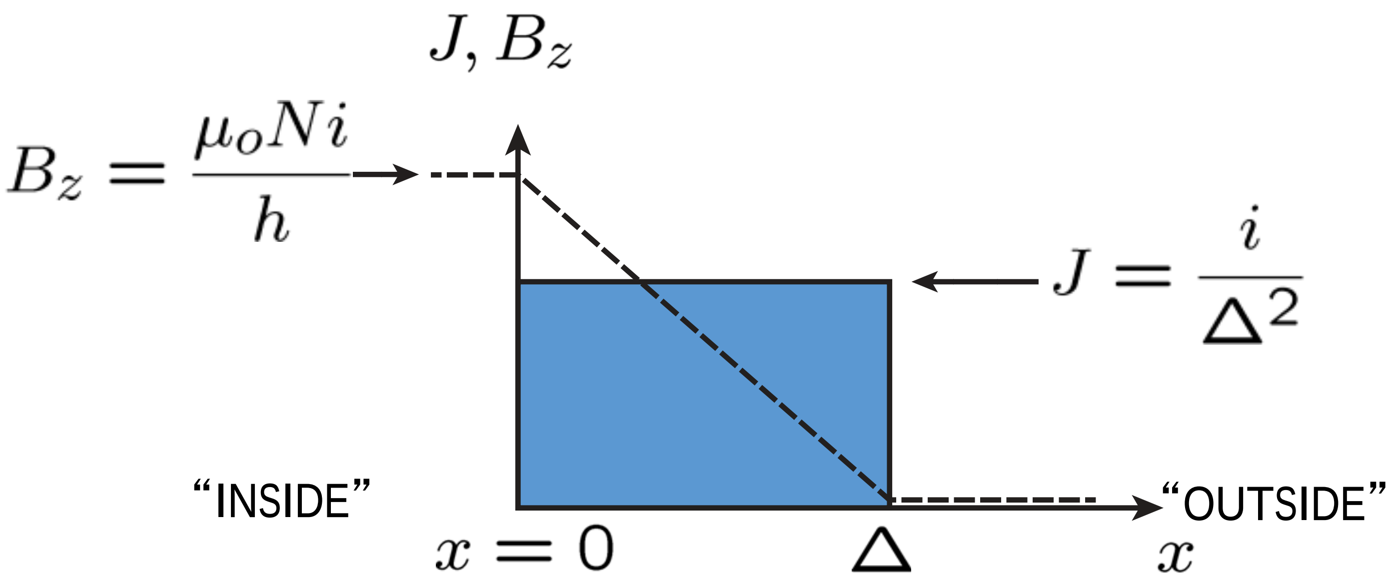
Current density inside each wire...

$$J = \frac{i}{\Delta^2}$$

...how does the field vary across wire ?

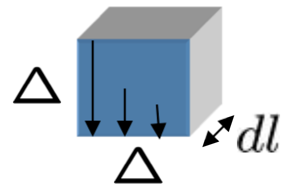
The H-fields Inside the Solenoid Wire

H-field changes linearly from inside to outside of the wire…



Force on the Incremental Section of the Solenoid Wire

$$\vec{F} = (\vec{J} \times \vec{B})$$

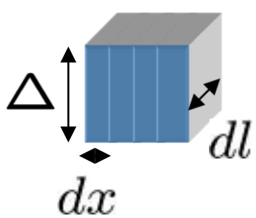


The incremental **force density** acting on the small 'cube' of wire is...

$$\vec{F} = \boxed{}$$

The incremental **force** acting on the small 'cube' of wire is...

$$\frac{df}{dl} = \Delta \int_0^{\Delta} F \cdot dx$$



$$\frac{df}{dl} = \boxed{}$$

Force on the Solenoid

$$f = \int \frac{df}{dl} dl =$$

For N wires, the total radial force on the solenoid is…

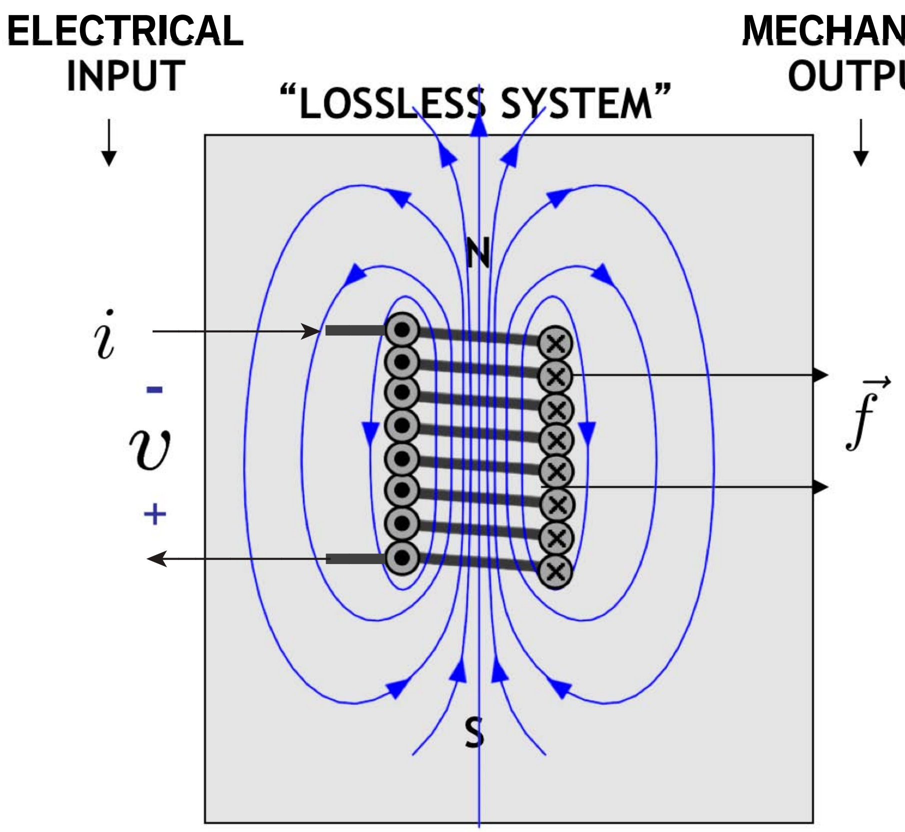
$$f_r = N f_w = \frac{\mu_o N^2 i^2}{h} \cdot \pi R$$

Note:

- Radial force grows as i^2 – does not depend on direction of current
- Tends to expand the coil

This was a relatively simple problem and it took a while to derive …
IS THERE AN EASIER WAY ?

Energy Method: An Easier Way !



First law for solenoid...

$$W_{electrial} = W_s + W_{mech}$$

Power flow...

$$i \cdot v = \frac{dW_s}{dt} + f_r \frac{dr}{dt}$$

How is energy stored in the coil ?

Reminder: Stored Energy in the (Linear) Coil

$$P_{elec} = v \cdot i = L\frac{di}{dt} \cdot i = \frac{1}{2}L\frac{d}{dt}i^2$$

If L is not a function of time…

$$P_{elec} = \frac{d}{dt}\left(\frac{1}{2}Li^2\right) = \frac{dW_s}{dt}$$

…where E is energy stored in the field of the inductor any instant in time

$$W_s(i,r) = \frac{1}{2}Li^2 = \frac{1}{2}\frac{\lambda^2}{L}$$

$$L = \frac{\mu_o N^2 A}{h}$$

Relating Stored Energy to Force

Lets use chain rule...

$$\frac{dW_s(\Phi, r)}{dt} = \frac{\partial W_s}{\partial \Phi}\frac{d\Phi}{dt} + \frac{\partial W_s}{\partial r}\frac{dr}{dt}$$

This looks familiar...

$$\frac{dW_s}{dt} = i \cdot v - f_r \frac{dr}{dt}$$

$$= iL\frac{di}{dt} - f_r \frac{dr}{dt}$$

Comparing similar terms suggests...

$$\boxed{f_r = -\frac{\partial W_s}{\partial r}}$$

Stored Energy and Force

Consider stored energy to depend on flux linkage and radius:

$$W_s = W_s(\lambda, R)$$

Lets use the chain rule...

q

$$\frac{d}{dt}W_s = \left.\frac{\partial W_s}{\partial \lambda}\right|_r \cdot \frac{d\lambda}{dt} + \left.\frac{\partial W_s}{\partial r}\right|_\lambda \cdot \frac{dr}{dt}$$

POWER BEING STORED

ELECTRICAL INPUT POWER

MECHANICAL OUTPUT POWER

$$\frac{dW_s}{dt} = i \cdot v - f_r \frac{dr}{dt}$$

VELOCITY

$$W_s(\lambda, R) = \frac{\lambda^2}{2L(R)}$$

$$\left.\frac{\partial W_s}{\partial \lambda}\right|_R = \frac{\lambda}{L} = i$$

STORED ENERGY FOR SHORTED SOLENOID

$$v = \frac{d\lambda}{dt}$$

VOLTAGE IN TERMS OF FLUX LINKAGE FOR SOLENOID

$$f = -\left.\frac{\partial W_s}{\partial R}\right|_\lambda$$

FORCE COMPUTED USING PARTIAL DERIVATIVE

Force on the Solenoid

If we can find the stored energy, we can immediately compute the force…

…lets take all the things we know to put this together…

$$f_r = -\frac{\partial W_s}{\partial r} \qquad W_s(\Phi, r) = \frac{1}{2}\frac{\Phi^2}{L} \qquad L = \frac{\mu_o N^2 \pi R^2}{h}$$

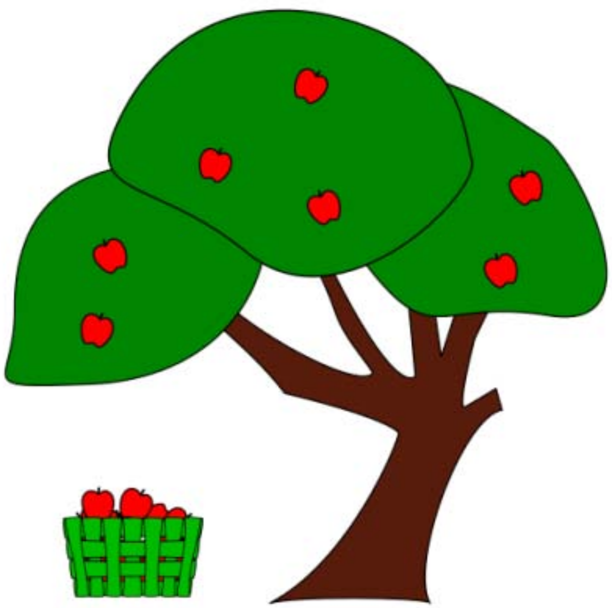
Exactly what we got from calculating internal fields and taking integrals over multiple dimensions

$$f_r = \boxed{\quad}$$

Force from potential and stored energy

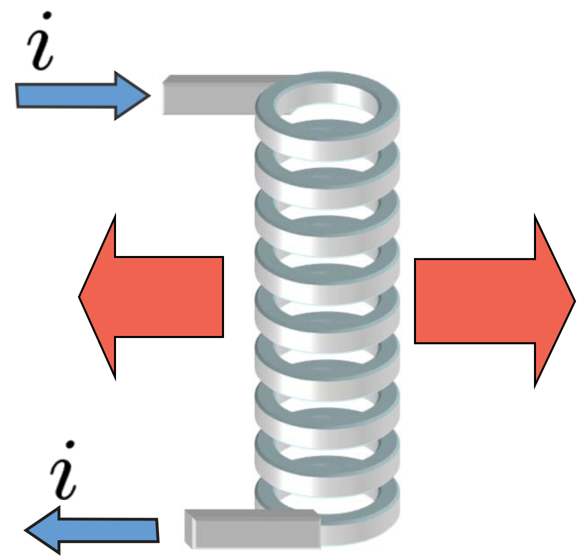
Force from a potential:

$$\vec{f} = -\nabla U$$

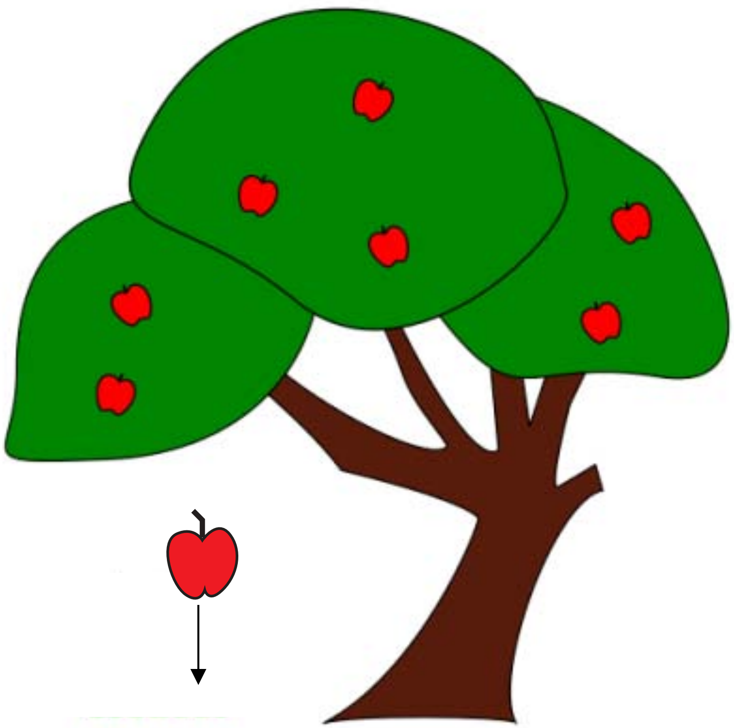


Force from stored energy:

$$\vec{f} = -\nabla W_s$$



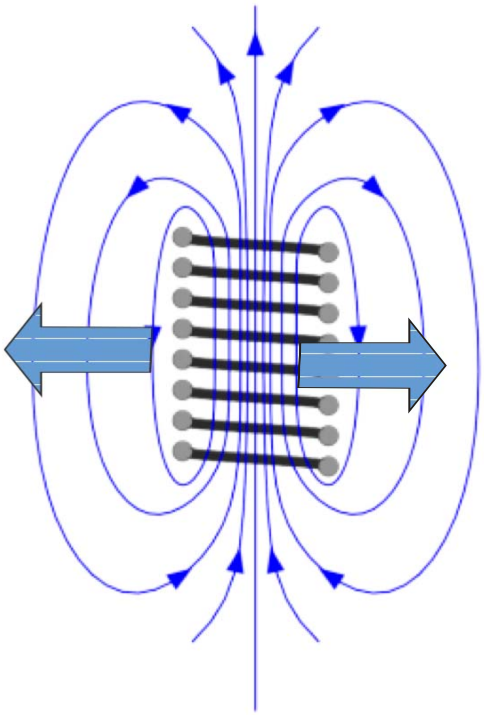
Gravitational force:



$$U(z) = mgz$$

$$\vec{f} = -\hat{z}\frac{d}{dz}U(z) = -\hat{z}mg$$

Magnetic Force



$$W_s(r) = \frac{\lambda^2}{2L(r)} = \frac{h\lambda^2}{2\mu_o N^2 \pi r^2}$$

$$\vec{f} = -\hat{r}\frac{d}{dr}W_s(r)$$

$$= -\hat{r}\frac{\mu_o N^2 i^2 \pi r}{h}$$

Energy Density of the Magnetic Field

What is the energy density stored in the coil ?

For a long coil the stored energy is...

$$\frac{W_s}{V} = \frac{\frac{1}{2}Li^2}{A \cdot h} = \frac{\frac{1}{2}\frac{\mu_o N^2 A}{h} i^2}{A \cdot h} = \frac{1}{2}\frac{\mu_o N^2 i^2}{h^2}$$

We can rewrite this as

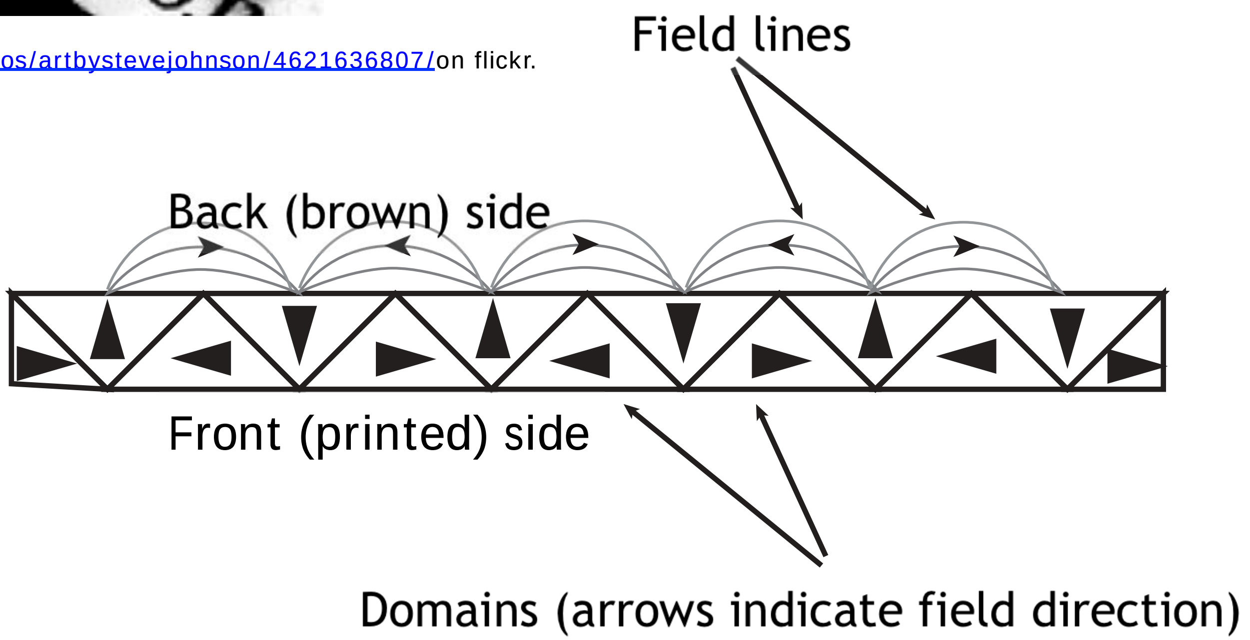
$$\frac{W_s}{V} = \frac{1}{2}\frac{\mu_o N^2 i^2}{h} = \frac{1}{2}\mu_o H \cdot H$$

The magnetic field not only generates a force, but can also be used to find the stored energy !

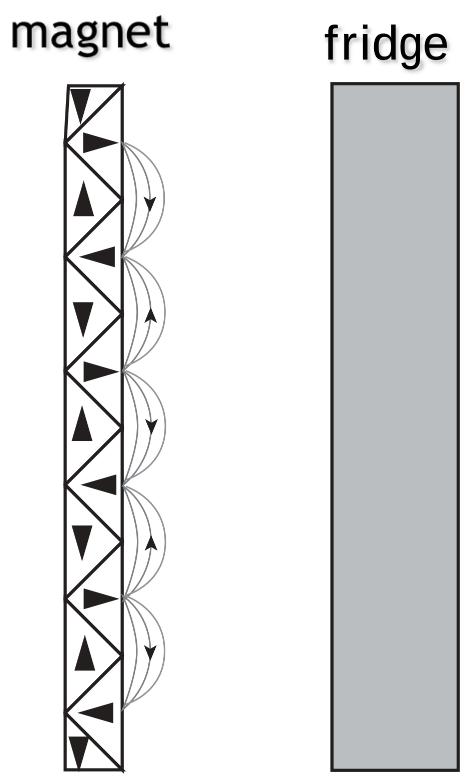
Magnetic Poetry

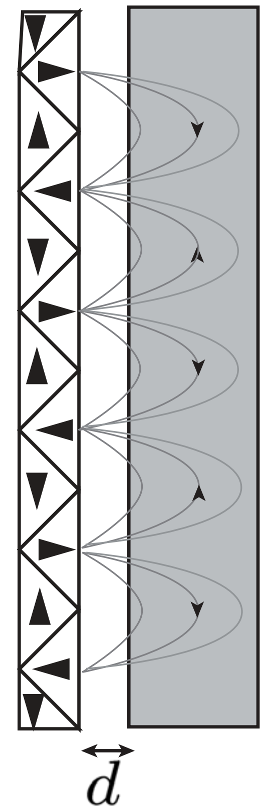
Image by Steve Johnson
http://www.flickr.com/photos/artbystevejohnson/4621636807/on flickr.

magnetic strontium ferrite, SrFe12O19, particles dispersed in the elastomer Hypalon



Magnetic Poetry





$$f_r = -\frac{\partial W_s}{\partial d}$$

$$\frac{W_s}{V} = \frac{1}{2}\mu_o H_{gap}^2$$

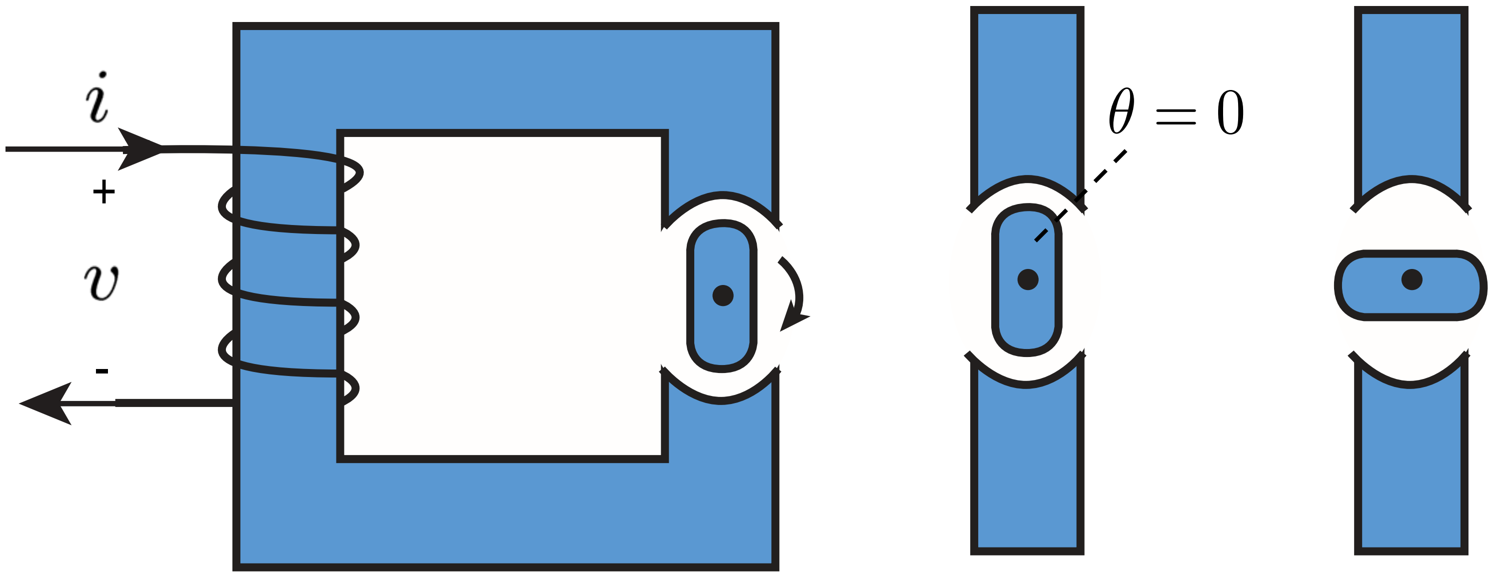
$$W_s = (Ad)\frac{1}{2}\frac{B_{gap}^2}{\mu_o}$$

$$\underbrace{\frac{f}{A}}_{\text{PRESSURE}} = \frac{B_{gap}^2}{2\mu_o}$$

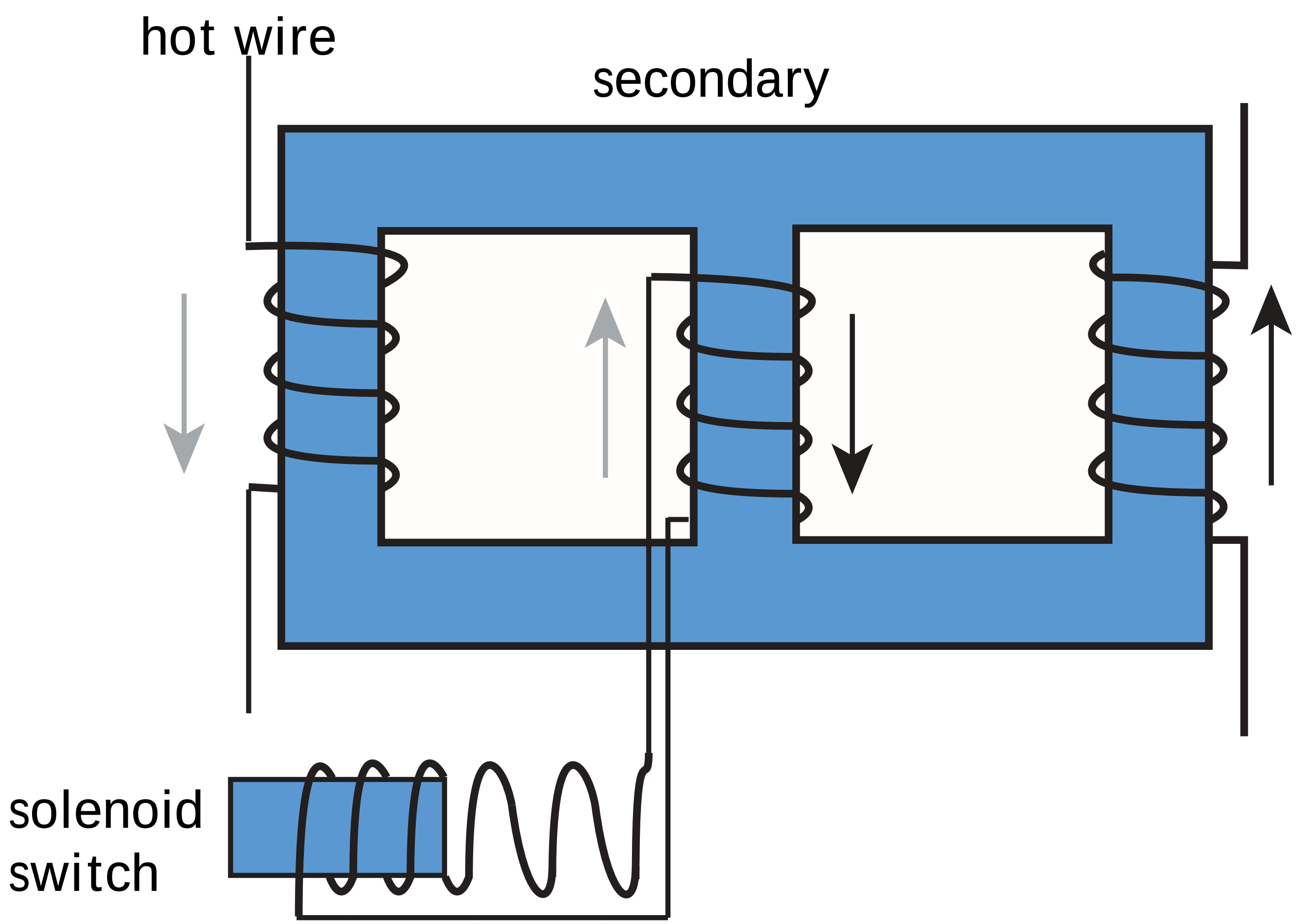
Magnetic Circuit Example

Variable-Reluctance Motor

Which configuration has the max flux, λ ?

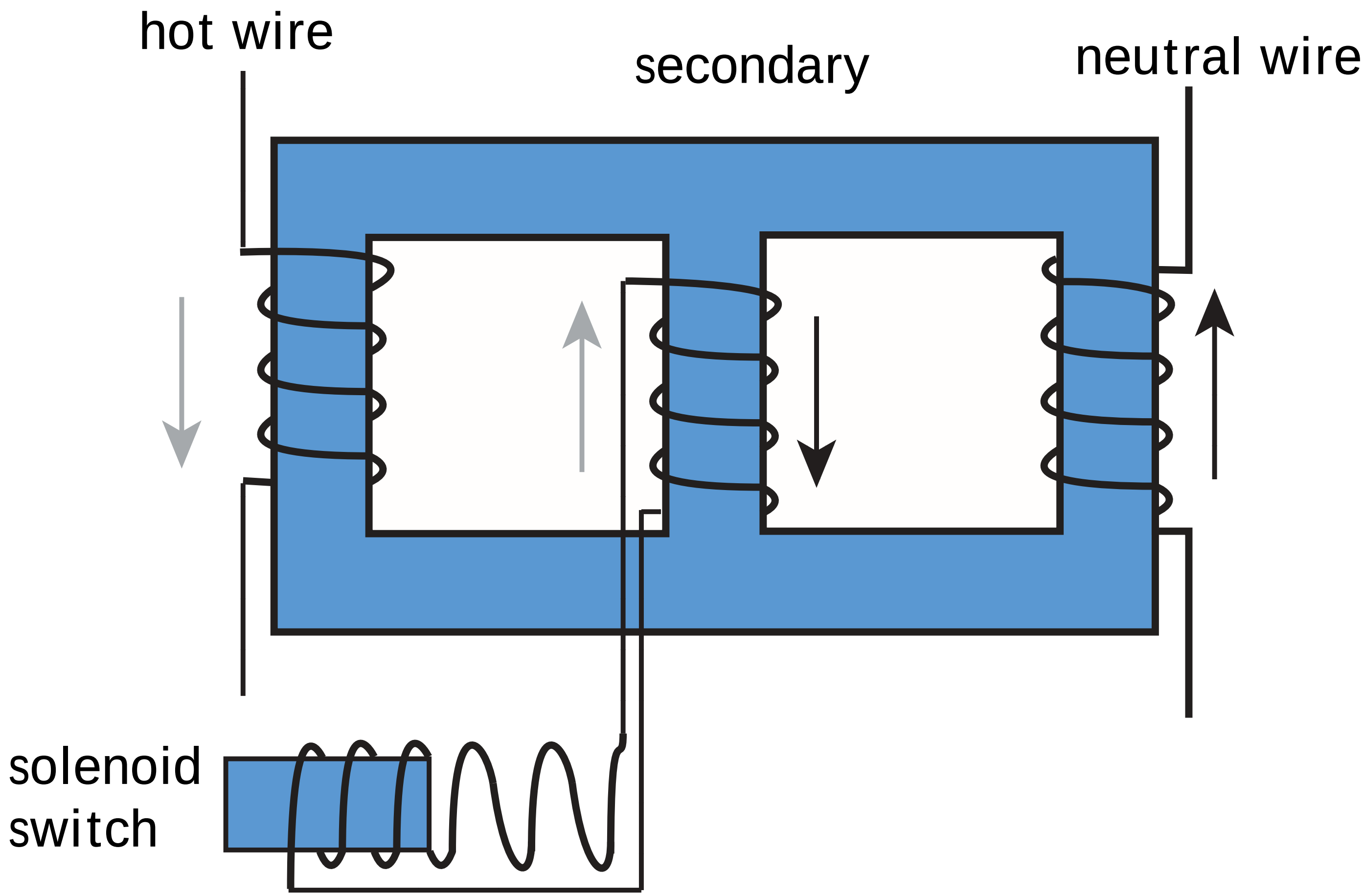


Example: Differential Transformer



If the current in the hot wire is the same as the current in the neutral wire, the induced current in the secondary is zero.

Example: Differential Transformer



If some current is lost,

current in the secondary opens the solenoid switch.

KEY TAKEAWAYS

Energy method for calculating Forces
calculated at constant flux linkage

$$f_r = -\frac{\partial W_s}{\partial r}$$

6.007 Electromagnetic Energy: From Motors to Lasers
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