Lecture 18: Pseudopolynomial

Dynamic Programming Steps (SRT BOT)

1. **Subproblem** definition subproblem $x \in X$
   - Describe the meaning of a subproblem **in words**, in terms of parameters
   - Often subsets of input: prefixes, suffixes, contiguous substrings of a sequence
   - Often multiply possible subsets across multiple inputs
   - Often record partial state: add subproblems by incrementing some auxiliary variables
   - Often smaller integers than a given integer (**today's focus**)
2. **Relate** subproblem solutions recursively $x(i) = f(x(j), \ldots)$ for one or more $j < i$
   - Identify a question about a subproblem solution that, if you knew the answer to, reduces the subproblem to smaller subproblem(s)
   - Locally brute-force all possible answers to the question
3. **Topological order** to argue relation is acyclic and subproblems form a DAG
4. **Base** cases
   - State solutions for all (reachable) independent subproblems where relation breaks down
5. **Original problem**
   - Show how to compute solution to original problem from solutions to subproblem(s)
   - Possibly use parent pointers to recover actual solution, not just objective function
6. **Time** analysis
   - $\sum_{x \in X} \text{work}(x)$, or if $\text{work}(x) = O(W)$ for all $x \in X$, then $|X| \cdot O(W)$
   - $\text{work}(x)$ measures **nonrecursive** work in relation; treat recursions as taking $O(1)$ time

Rod Cutting

- Given a rod of length L and value $v(\ell)$ of rod of length ℓ for all $\ell \in \{1, 2, \ldots, L\}$
- Goal: Cut the rod to maximize the value of cut rod pieces
- Example: $L = 7$, $v \underset{\ell =}{=} [\underset{0}{0}, \underset{1}{1}, \underset{2}{10}, \underset{3}{13}, \underset{4}{18}, \underset{5}{20}, \underset{6}{31}, \underset{7}{32}]$
- Maybe greedily take most valuable per unit length?
- Nope! $\arg\max_\ell v[\ell]/\ell = 6$, and partitioning $[6, 1]$ yields 32 which is not optimal!
- Solution: $v[2] + v[2] + v[3] = 10 + 10 + 13 = 33$
- Maximization problem on value of partition

1. **Subproblems**
   - $\boxed{x(\ell)\text{: maximum value obtainable by cutting rod of length } \ell}$
   - For $\ell \in \{0, 1, \ldots, L\}$
2. **Relate**
   - First piece has some length p **(Guess!)**
   - $x(\ell) = \max\{v(p) + x(\ell \quad p) \mid p \in \{1, \ldots, \ell\}\}$
   - (draw dependency graph)
3. **Topological order**
   - Increasing ℓ: Subproblems $x(\ell)$ depend only on strictly smaller ℓ, so acyclic
4. **Base**
   - $x(0) = 0$ (length-zero rod has no value!)
5. **Original problem**
   - Maximum value obtainable by cutting rod of length L is $x(L)$
   - Store choices to reconstruct cuts
   - If current rod length ℓ and optimal choice is ℓ', remainder is piece $p = \ell \quad \ell'$
   - (maximum-weight path in subproblem DAG!)
6. **Time**
   - # subproblems: $L + 1$
   - work per subproblem: $O(\ell) = O(L)$
   - $O(L^2)$ running time

Is This Polynomial Time?

- **(Strongly) polynomial time** means that the running time is bounded above by a constant-degree polynomial in the **input size** measured in words
- In Rod Cutting, input size is $L + 1$ words (one integer L and L integers in v)
- $O(L^2)$ is a constant-degree polynomial in $L + 1$, so YES: (strongly) polynomial time

```
# recursive
x = {}
def cut_rod(l, v):
    if l < 1:   return 0                      # base case
    if l not in x:                            # check memo
        for piece in range(1, l + 1):         # try piece
            x_ = v[piece] + cut_rod(l - piece, v)   # recurrence
            if (l not in x) or (x[l] < x_):   # update memo
                x[l] = x_
    return x[l]
```

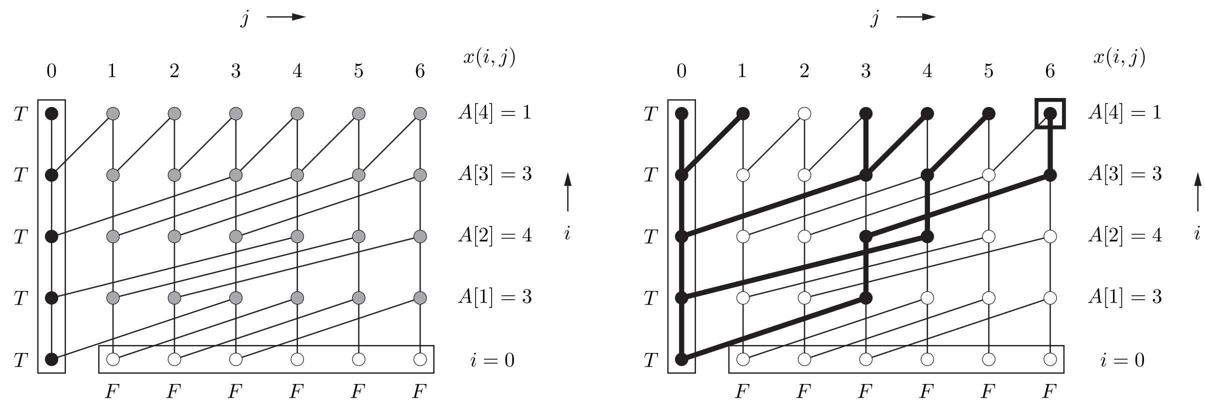
```
# iterative
def cut_rod(L, v):
    x = [0] * (L + 1)                         # base case
    for l in range(L + 1):                    # topological order
        for piece in range(1, l + 1):         # try piece
            x_ = v[piece] + x[l - piece]      # recurrence
            if x[l] < x_:                     # update memo
                x[l] = x_
    return x[L]
```

```
# iterative with parent pointers
def cut_rod_pieces(L, v):
    x = [0] * (L + 1)                         # base case
    parent = [None] * (L + 1)                 # parent pointers
    for l in range(1, L + 1):                 # topological order
        for piece in range(1, l + 1):         # try piece
            x_ = v[piece] + x[l - piece]      # recurrence
            if x[l] < x_:                     # update memo
                x[l] = x_
                parent[l] = l - piece         # update parent
    l, pieces = L, []
    while parent[l] is not None:              # walk back through parents
        piece = l - parent[l]
        pieces.append(piece)
        l = parent[l]
    return pieces
```

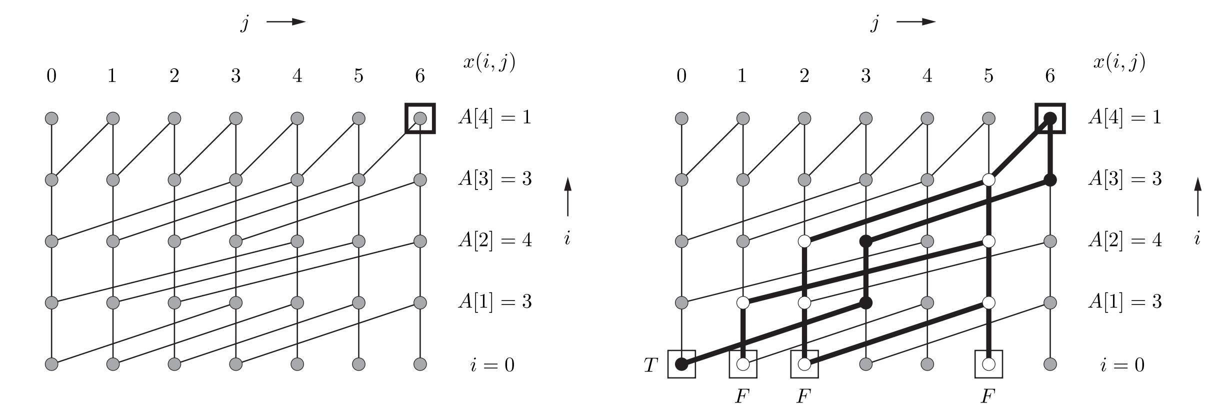
Subset Sum

- Input: Sequence of n positive integers $A = \{a_0, a_1, \ldots, a_n\}$
- Output: Is there a subset of A that sums exactly to T? (i.e., $\exists A' \subseteq A$ s.t. $\sum_{a \in A'} a = T$?)
- Example: $A = (1, 3, 4, 12, 19, 21, 22)$, $T = 47$ allows $A' = \{3, 4, 19, 21\}$
- Optimization problem? Decision problem! Answer is YES or NO, TRUE or FALSE
- In example, answer is YES. However, answer is NO for some T, e.g., $2, 6, 9, 10, 11, \ldots$

1. **Subproblems**
   - $\boxed{x(i, t) = \text{does any subset of } A[i:] \text{ sum to } t\text{?}}$
   - For $i \in \{0, 1, \ldots, n\}, t \in \{0, 1, \ldots, T\}$
2. **Relate**
   - Idea: Is first item a_i in a valid subset A'? (Guess!)
   - If yes, then try to sum to $t \quad a_i \quad 0$ using remaining items
   - If no, then try to sum to t using remaining items
   - $x(i, t) = \text{OR} \begin{cases} x(i+1, t \quad A[i]) & \text{if } t \quad A[i] \\ x(i+1, t) & \text{always} \end{cases}$
3. **Topological order**
   - Subproblems $x(i, t)$ only depend on strictly larger i, so acyclic
   - Solve in order of decreasing i
4. **Base**
   - $x(i, 0) = \text{YES}$ for $i \in \{0, \ldots, n\}$ (space packed exactly!)
   - $x(0, t) = \text{NO}$ for $j \in \{1, \ldots, T\}$ (no more items available to pack)
5. **Original problem**
   - Original problem given by $x(0, T)$
   - Example: $A = (3, 4, 3, 1), T = 6$ solution: $A' = (3, 3)$
   - Bottom up: Solve all subproblems (Example has 35)



- Top down: Solve only **reachable** subproblems (Example, only 14!)



6. **Time**

- # subproblems: $O(nT)$, $O(1)$ work per subproblem, $O(nT)$ time

Is This Polynomial?

- Input size is $n + 1$: one integer T and n integers in A
- Is $O(nT)$ bounded above by a polynomial in $n + 1$? NO, not necessarily
- On w-bit word RAM, $T \leq 2^w$ and w $\lg(n + 1)$, but we don't have an upper bound on w
- E.g., $w = n$ is not unreasonable, but then running time is $O(n2^n)$, which is **exponential**

Pseudopolynomial

- Algorithm has **pseudopolynomial time**: running time is bounded above by a constant-degree polynomial in input size and input integers
- Such algorithms are polynomial in the case that integers are polynomially bounded in input size, i.e., $n^{O(1)}$ (same case that Radix Sort runs in $O(n)$ time)
- Counting sort $O(n + u)$, radix sort $O(n \log_n u)$, direct-access array build $O(n + u)$, and Fibonacci $O(n)$ are all pseudopolynomial algorithms we've seen already
- Radix sort is actually **weakly polynomial** (a notion in between strongly polynomial and pseudopolynomial): bounded above by a constant-degree polynomial in the input size measured in bits, i.e., in the logarithm of the input integers
- Contrast with Rod Cutting, which was polynomial
  - Had pseudopolynomial dependence on L
  - But luckily had L input integers too
  - If only given subset of sellable rod lengths (Knapsack Problem, which generalizes Rod Cutting and Subset Sum — see recitation), then algorithm would have been only pseudopolynomial

Complexity

- Is Subset Sum solvable in polynomial time when integers are not polynomially bounded?
- No if P $\neq$ NP. What does that mean? Next lecture!

Main Features of Dynamic Programs

- Review of examples from lecture
- **Subproblems:**
  - **Prefix/suffixes:** Bowling, LCS, LIS, Floyd–Warshall, Rod Cutting (coincidentally, really Integer subproblems), Subset Sum
  - **Substrings:** Alternating Coin Game, Arithmetic Parenthesization
  - **Multiple sequences:** LCS
  - **Integers:** Fibonacci, Rod Cutting, Subset Sum
    - **Pseudopolynomial:** Fibonacci, Subset Sum
  - **Vertices:** DAG shortest paths, Bellman–Ford, Floyd–Warshall
- **Subproblem constraints/expansion:**
  - **Nonexpansive constraint:** LIS (include first item)
  - **$2\times$ expansion:** Alternating Coin Game (who goes first?), Arithmetic Parenthesization (min/max)
  - **$\Theta(1)\times$ expansion:** Piano Fingering (first finger assignment)
  - **$\Theta(n)\times$ expansion:** Bellman–Ford (# edges)
- **Relation:**
  - **Branching** = # dependant subproblems in each subproblem
  - **$\Theta(1)$ branching:** Fibonacci, Bowling, LCS, Alternating Coin Game, Floyd–Warshall, Subset Sum
  - **$\Theta(\text{degree})$ branching** (source of $|E|$ in running time): DAG shortest paths, Bellman–Ford
  - **$\Theta(n)$ branching:** LIS, Arithmetic Parenthesization, Rod Cutting
  - **Combine multiple solutions (not path in subproblem DAG):** Fibonacci, Floyd–Warshall, Arithmetic Parenthesization
- **Original problem:**
  - **Combine multiple subproblems:** DAG shortest paths, Bellman–Ford, Floyd–Warshall, LIS, Piano Fingering