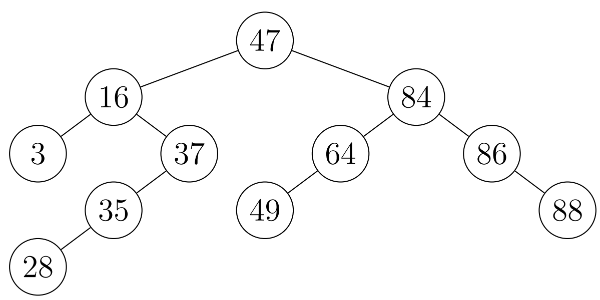
Problem Set 4

Please write your solutions in the LaTeX and Python templates provided. Aim for concise solutions; convoluted and obtuse descriptions might receive low marks, even when they are correct.

Problem 4-1. [10 points] **Binary Tree Practice**

(a) [2 points] The Set Binary Tree `T` below is **not height-balanced** but does satisfy the **binary search tree** property, assuming the key of each integer item is itself. Indicate the keys of all nodes that are not height-balanced and compute their skew.



(b) [5 points] Perform the following insertions and deletions, one after another in sequence on `T`, by adding or removing a leaf while maintaining the binary search tree property (a key may need to be swapped down into a leaf). For this part, **do not** use rotations to balance the tree. Draw the modified tree after each operation.

```
T.insert(2)
T.delete(49)
T.delete(35)
T.insert(85)
T.delete(84)
```

(c) [3 points] For each unbalanced node identified in part (a), draw the two trees that result from rotating the node in the **original** tree left and right (when possible). For each tree drawn, specify whether it is height-balanced, i.e., all nodes satisfy the AVL property.

Note: Material on this page requires material that will be covered in **L08 on March 3, 2020**. We suggest waiting to solve these problem until after that lecture. All other pages of this assignment can be solved using only material from L07 and earlier.

Problem 4-2. Heap Practice [10 points]

For each array below, draw it as a **complete**[1] binary tree and state whether the tree is a max-heap, a min-heap, or neither. If the tree is neither, turn the tree into a min-heap by repeatedly swapping items that are **adjacent in the tree**. Communicate your swaps by drawing a sequence of trees, marking on each tree the pair that was swapped.

(a) `[4, 12, 8, 21, 14, 9, 17]`

(b) `[701, 253, 24, 229, 17, 22]`

(c) `[2, 9, 13, 8, 0, 2]`

(d) `[1, 3, 6, 5, 4, 9, 7]`

Problem 4-3. [10 points] **Gardening Contest**

Gardening company Wonder-Grow sponsors a nation-wide gardening contest each year where they rate gardens around the country with a positive integer[2] ***score***. A garden is designated by a ***garden pair*** (s_i, r_i), where s_i is the garden's assigned score and r_i is the garden's unique positive integer ***registration number***.

(a) [5 points] To support inclusion and reduce competition, Wonder-Grow wants to award identical trophies to the top k gardens. Given an unsorted array A of garden pairs and a positive integer $k \leq |A|$, describe an $O(|A| + k \log |A|)$-time algorithm to return the registration numbers of k gardens in A with highest scores, breaking ties arbitrarily.

(b) [5 points] Wonder-Grow decides to be more objective and award a trophy to every garden receiving a score strictly greater than a reference score x. Given a max-heap A of garden pairs, describe an $O(n_x)$-time algorithm to return the registration numbers of all gardens with score larger than x, where n_x is the number of gardens returned.

[1] Recall from Lecture 8 that a binary tree is ***complete*** if it has exactly 2^i nodes of depth i for all i except possibly the largest, and at the largest depth, all nodes are as far left as possible.

[2] In this class, when an integer or string appears in an input, without listing an explicit bound on its size, you should assume that it is provided inside a constant number of machine words in the input.

Problem 4-4. [15 points] **Solar Supply**

Entrepreneur Bonty Murns owns a set S of n solar farms in the town of Fallmeadow. Each solar farm $(s_i, c_i) \in S$ is designated by a unique positive integer ***address*** s_i and a farm ***capacity*** c_i: a positive integer corresponding to the maximum energy production rate the farm can support. Many buildings in Fallmeadow want power. A building (b_j, d_j) is designated by a unique ***name*** string b_j and a ***demand*** d_j: a positive integer corresponding to the building's energy consumption rate.

To receive power, a building in Fallmeadow must be connected to a **single** solar farm under the restriction that, for any solar farm s_i, the sum of demand from all the buildings connected to s_i may not exceed the farm's capacity c_i. Describe a database supporting the following operations, and for each operation, specify whether your running time is worst-case, expected, and/or amortized.

`initialize(`S`)`	Initialize database with a list $S = ((s_0, c_0), \ldots, (s_{n-1}, c_{n-1}))$ corresponding to n solar farms in $O(n)$ time.
`power_on(`b_j`,`d_j`)`	Connect a building with name b_j and demand d_j to any solar farm having available capacity at least d_j in $O(\log n)$ time (or return that no such solar farm exists).
`power_off(`b_j`)`	Remove power from the building with name b_j in $O(\log n)$ time.
`customers(`s_i`)`	Return the names of all buildings supplied by the farm at address s_i in $O(k)$ time, where k is the number of building names returned.

Problem 4-5. [15 points] **Robot Wrangling**

Dr. Squid has built a robotic arm from $n+1$ rigid bars called ***links***, each connected to the one before it with a rotating joint (n joints in total). Following standard practice in robotics[3], the orientation of each link is specified locally relative to the orientation of the previous link. In mathematical notation, the change in orientation at a joint can be specified using a 4×4 ***transformation matrix***. Let $\mathcal{M} = (M_0, \ldots, M_{n-1})$ be an array of transformation matrices associated with the arm, where matrix M_k is the change in orientation at joint k, between links k and $k+1$.

To compute the position of the ***end effector***[4], Dr. Squid will need the arm's ***full transformation***: the ordered matrix product of the arm's transformation matrices, $\prod_{k=0}^{n-1} M_k = M_0 \cdot M_1 \cdot \ldots \cdot M_{n-1}$. Assume Dr. Squid has a function `matrix_multiply(`M_1`,`M_2`)` that returns the matrix product[5] $M_1 \times M_2$ of any two 4×4 transformation matrices in $O(1)$ time. While tinkering with the arm changing one joint at a time, Dr. Squid will need to re-evaluate this matrix product quickly. Describe a database to support the following **worst-case** operations to accelerate Dr. Squid's workflow:

`initialize(`$\mathcal{M}$`)`	Initialize from an initial input configuration $\mathcal{M}$ in $O(n)$ time.
`update_joint(`k`,`M`)`	Replace joint k's matrix M_k with matrix M in $O(\log n)$ time.
`full_transformation()`	Return the arm's current full transformation in $O(1)$ time.

[3] More on forward kinematic robotics computation here: https://en.wikipedia.org/wiki/Forward_kinematics

[4] i.e., the device at the end of a robotic arm: https://en.wikipedia.org/wiki/Robot_end_effector

[5] Recall, matrix multiplication is not commutative, i.e., $M_1 \cdot M_2 \neq M_2 \cdot M_1$, except in very special circumstances.

Problem 4-6. [40 points] **$\pi z^2 a$ Optimization**

Liza Pover has found a Monominos pizza left over from some big-TEX recruiting event. The pizza is a disc[6] with radius z, having n ***toppings*** labeled $0, \ldots, n-1$. Assume z fits in a single machine word, so integer arithmetic on $O(1)$ such integers can be done in $O(1)$ time. Each topping i:

- is located at Cartesian coordinates (x_i, y_i) where x_i, y_i are integers from range $R = \{-z, \ldots, z\}$ (you may assume that **all coordinates are distinct**), and
- has integer ***tastiness*** $t_i \in R$ (note, topping tastiness can be negative, e.g., if it's pineapple[7]).

Liza wants to pick a point (x', y') and make a pair of cuts from that point, one going straight down and one going straight left, and take the resulting ***slice***, i.e., the intersection of the pizza with the two half-planes $x \leq x'$ and $y \leq y'$. The tastiness of this slice is the sum of all t_i such that $x_i \leq x'$ and $y_i \leq y'$. Liza wants to find a ***tastiest*** slice, that is, a slice of maximum tastiness. Assume there exists a slice with **positive tastiness**.

(a) [2 points] If point (x', y') results in a slice with tastiness $t \neq 0$, show there exists $i, j \in \{0, 1, \ldots, n-1\}$ such that point (x_i, y_j) results in a slice of equal tastiness t (i.e., a tastiest slice exists resulting from a point that is both vertically and horizontally aligned with toppings).

(b) [8 points] To make finding a tastiest slice easier, show how to modify a Set AVL Tree so that:

- it stores **key–value items**, where each item x contains a value $x.val$ (in addition to its key $x.key$ on which the Set AVL is ordered);
- it supports a new tree-level operation `max prefix()` which returns in **worst-case** $O(1)$ time a pair $(k^*, \text{prefix}(k^*))$, where k^* is any key stored in the tree T that maximizes the ***prefix sum***, $\text{prefix}(k) = \{x.val \mid x \in T \text{ and } x.key \leq k\}$ (that is, the sum of all values of items whose keys are $\leq k$); and
- all other Set AVL Tree operations maintain their running times.

(c) [5 points] Using the data structure from part (b) as a black box, describe a **worst-case** $O(n \log n)$-time algorithm to return a triple (x, y, t), where point (x, y) corresponds to a slice of maximum tastiness t.

(d) [25 points] Write a Python function `tastiest_slice(toppings)` that implements your algorithm from part (c), including an implementation of your data structure from part (b).

[6] The pizza has thickness a, so it has volume $\pi z^2 a$.

[7] If you believe that Liza's Pizza preferences are objectively wrong, feel free to assert your opinions on Piazza.

```python
from Set_AVL_Tree import BST_Node, Set_AVL_Tree

class Key_Val_Item:
    def __init__(self, key, val):
        self.key = key
        self.val = val

    def __str__(self):
        return "%s,%s" % (self.key, self.val)

class Part_B_Node(BST_Node):
    def subtree_update(A):
        super().subtree_update()
        #########################################
        # ADD ANY NEW SUBTREE AUGMENTATION HERE #
        #########################################

class Part_B_Tree(Set_AVL_Tree):
    def __init__(self):
        super().__init__(Part_B_Node)

    def max_prefix(self):
        '''
        Output: (k, s) | a key k stored in tree whose
                       | prefix sum s is maximum
        '''
        k, s = 0, 0
        ##################
        # YOUR CODE HERE #
        ##################
        return (k, s)

def tastiest_slice(toppings):
    '''
    Input:  toppings | List of integer tuples (x,y,t) representing
                     | a topping at (x,y) with tastiness t
    Output: tastiest | Tuple (X,Y,T) representing a tastiest slice
                     | at (X,Y) with tastiness T
    '''
    B = Part_B_Tree()   # use data structure from part (b)
    X, Y, T = 0, 0, 0
    ##################
    # YOUR CODE HERE #
    ##################
    return (X, Y, T)
```