MIT OpenCourseWare
http://ocw.mit.edu

6.002 Circuits and Electronics, Spring 2007

Please use the following citation format:

> Anant Agarwal, *6.002 Circuits and Electronics, Spring 2007* (Massachusetts Institute of Technology: MIT OpenCourseWare). http://ocw.mit.edu (accessed MM DD, YYYY). License: Creative Commons Attribution-Noncommercial-Share Alike.

Note: Please use the actual date you accessed this material in your citation.

For more information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms

MIT OpenCourseWare
http://ocw.mit.edu

6.002 Circuits and Electronics, Spring 2007
Transcript – Lecture 17

Good morning. All right. Today we are going to take a fresh look at some of the stuff we covered in the last two lectures. And the graph I want you to keep in mind as we go through this lecture in terms of what to expect.

This was time. And last Tuesday's lecture we covered some stuff. I talked about a method for the sinusoidal response which was agony, I warned you it will be agony, and then towards the end I showed you another method that was quite a bit easier but still pretty hard.

And I promised you that today there will be a new method which is going to be so easy , actually almost trite. Just imagine. I am going to make a statement right now that I think you will all find hard to believe.

What I am going to say is just imagine your RLC circuit, your resistor, inductor and capacitor, a parallel form or series form. Imagine that you could write down the characteristic equation for that by observation in 30 seconds or less.

Just imagine that. By observation, boom, write down the characteristic equation for virtually any RLC circuit or RC circuit or whatever. And we all know that once you have the characteristic equation you could very easily go from there to the time domain response intuitively or to the sinusoidal steady-state response, too.

So just keep that thought in mind. Imagine 30 seconds. And that is what you should expect in today's lecture. Students often ask me, if this stuff is actually so easy why do you take us through this tortuous path? Are we just mean? Do we just want you show you how hard things are and then show the easy way? I have argued with myself every year as to whether to just go ahead and give the easy path and that's it.

But I think the reason we cover the basic foundations is that it gives you a level of insight that you would not have otherwise gotten if I directly jumped into the easy method. So you need to understand the foundations and you need to have seen that at least once.

And second, once you do something the hard way, you appreciate all the more the easy method. All right. Today we cover what is called "The Impedance Model". First let me do a review just because of the large amount of content in the last two lectures.

I did them using view graphs. I usually don't like to do that, but even then it was quite rushed. So let me quickly summarize for you kind of the main points. We have been looking at, on Tuesday, the sinusoidal -- --looking at the sinusoidal steady state response.

Also fondly denoted as SSS. And the readings for this were Chapters 14.1 and 14.2. what we said was if you took this example circuit and we fed as input cosine of omega t, we have an R and a C, and let's say we cared about the output response and we cared about the capacitor voltage.

What we talked about was focused on the sinusoidal steady-state response. And what that meant was first of all focus on steady-state. In other words, just to capture the steady-state behavior when t goes to infinity after a long period of time.

And for most of the circuits that we consider, because of the R or presence of any resistance, the homogenous response usually would die out because the homogenous response is usually of the form minus t by tau.

And as t goes to infinity this term tends to go to zero. We are just looking at the steady-state. And therefore, because of the circuits we looked at, we can ignore the homogenous response. All we are left to do is to find the particular response to sinusoids of this form.

And second was focus on sinusoids. We said the reason for this was that, let's say we did not care particularly What happened when I just turned on my amplifier. I just turned on my amplifier, often times you see some distorted sound coming out for a few seconds and then hear a much clearer sound.

And that initial part is due to the transient response. And let's say we don't care about that. We care about the steady state. Second we focus on sinusoids because based on the Fourier series experience that you had previously, we can represent repeated signals as a sum of sines.

And therefore it is important to understand the behavior of these circuits when the input is a sinusoid. And what was important was this introduced a new way of looking at circuits, and that was the frequency viewpoint.

When we looked at transient responses, we plotted response as a function of time. And when we look at sinusoidal steady-state, it becomes interesting to plot the response as a function of the frequency, a function of omega.

What I will do is draw a little chart for you to sort of visualize the various processes we have been going through. We can liken obtaining the sinusoidal steady-state response to following these steps.

Here is my input. What I did as a first step was fed my input to a usual circuit model. My elements were lumped elements, built the circuit and wrote down the VI relationship for the element. As a second step I set up the differential equation.

This was the first of four steps, set up a differential equation. And then the path that I took first was fraught with real nightmarish trig. By the end of the day it would still yield an answer. It could be a nightmare.

But I would get something cosine omega t plus something, some phase. I could grunge through the trig. And I gave up halfway in class here, but you could grunge through it if you would like. And you would get the answer to be some sinusoid with some amplitude and some phase.

So Vi cosine omega t would produce the response that was something cosine omega t plus some phase. We said this was too painful so let's punt this. Instead, what we said we would do is take a detour, take an easier path.

And the easier path looked like this. I said let's sneak in -- -- Vie^(j omega t) drive. That is just imagine, do the math as if you had fed in not a Vi cosine omega t but a Vie^(j omega t). And from Euler's relation you know that the real part is Vi cosine omega t.

So we said that I am going to sneak in this thing, find the response and just take the real part of that because the real part of the input gives me this. So this is my "sneaky path". And what I did there, as soon as we fed in the e^(j omega t), because of the property of exponentials, the e^(j omega t) cancelled out in my equation.

And what was left was some fairly simple complex algebra. And at the end of the day, after I grunged through some fairly simple complex algebra, I ended up with some response that looked like this.

Vpe^(j omega t). What I would find is that for the input Vie^(j omega t), I would get a response Vpe^(j omega t). And then what I said we would do is take the real part. Why take the real part? Because this is a fake, a sneaky input.

The input I really care about is the real part of the sneaky input. So this is my sneaky output. And what I care about is the real part of the sneaky output. That is sort of the inverse superposition argument that I made on Tuesday that if what I care about is the real part of this input, then I just take the real part and get the output that I care about.

So I take the real part. Notice that Vp here, in the examples we did, we did an RC example. The Vp here was a complex number. So I could represent that complex number as, in many ways. This is e^(j omega t).

I could represent Vp in an amplitude, as a phasor, actually polar coordinates. I can say that the equivalent to Vpe to the j angle Vp. Vp is a complex number. If you look at the complex appendix in your course notes, I can represent a complex number as an amplitude multiplied by e raised to j times some phase.

It's simple complex algebra. And then what I could do here is take the real part of that. And when I took the real part of that what came about was that this was simply Vp. Notice that the angle Vp goes in here so it becomes j times omega t plus angle Vp.

It is Vp amplitude times e raised to j omega t plus j angle Vp. And the real part of that is simply Vp cosine of that stuff. What I end up getting here is Vp cosine omega t plus Vp. The cool thing to notice was that once I found out this response here, I could immediately write down the output based on Vp.

In other words, once I had Vp, I could stop right there in my math. I got Vp very quickly here. This step produced Vp very quickly, after two algebraic steps. And then from here I could directly write down the answer as homogenous of Vp cosine omega t plus angle Vp.

Boom, right there. So this was a much shorter path. And here I just described to you how this yields an expression for Vp and angle Vp. And for our example Vp was 1/(1+j omega RC). And we often times write a shorthand notation 1+sRC, where S is simply j omega.

We commonly jump back and forth between the shorthand notation S and j omega. S has some other fundamental, has another fundamental significance you will learn about in future courses, but for now S is simply a short form for j omega.

This was the path that we took. There is a hard path and an easier path. Today I am going to claim that even this was too hard. There is an even easier path. And today what I am going to show you is that from here we are going to take one step and get here.

I am going to show you today that we won't do this, we won't do this, not this, not this, none of this. One step and then we are going to get the answer. So let's do that. Before we jump into the impedance method and get into doing that, I just would like to plot for you this function here just so we can understand a little bit better exactly what is going on.

As I mentioned to you, the output vO for our circuit there was simply Vp cosine of omega T plus angle Vp. Oh, that's Vp so this one should be Vi here. I am showing you Vp so there is a Vi in there.

Vp/Vi=1/(1+j omega RC). This is a complex number, and it is simply a number that when multiplied with Vi gives me the output. This is also called a transfer function and represented as H(j omega). This guy is a transfer function, much like the gain of my amplifier.

Which when multiplied by the input to get me the output. This guy is a complex multiplier which when multiplied by Vi gives me Vp. And as such we call it a transfer function H(j omega). And we can plot this function.

Notice that this a function of omega. Remember we are taking the frequency domain view, so where has time vanished? Remember that we are taking the steady state view. So we are saying in the steady state, if I wait long enough this is how my circuit is going to behave, this is how a circuit is going to behave.

And the transient responses have died away and I have time in my output here so my output is a cosine. But that in itself is not very interesting. It is a cosine of some amplitude and has some phase.

What we will plot is we are going to plot this property here, Vp as a function of the frequency. Vp is frequency dependent. As an example, I could plot the absolute value of Vp/Vi, the modulus of that versus omega.

And notice that when omega is zero again intuitive ways of plotting this is to look at the value at zero and look at the value at large omega. For small omega, omega goes to zero this is one, so it starts off here.

And when omega is very large then it is much bigger than one here, so this goes down. Far away this one looks like 1/omega RC. And this function, assuming I have linear scales on my X and Y axes looks like this.

We also commonly plot this using log-log scales. And when you do log-log scales you get a straight line here, and then you actually get a straight line of slope minus one because the log of this gives you a line with a constant slope, it's a slope of negative one so it becomes a straight line going down.

The other interesting thing to realize is that this magnitude is simply one by one plus omega squared R squared C squared, the square root of this. That's the magnitude here. And notice when omega equals 1/RC, this thing, the denominator becomes one by square root of 2.

Somewhere here when omega equals 1/RC The output is one by square root 2 times the input. It's an interesting point. And this is called the "break frequency". You can view it as a frequency where I am getting this transition from one to a lower value, and it is where the output is one by square root two times the value of the input.

Now you can think back on the demo we showed you earlier. And in the demo remember that as I increased the frequency of my input sinusoid my output kept becoming smaller and smaller and smaller. And you notice that you can see this dying out or decaying of the amplitude as I increase my omega.

Let me go back. What you have done is that, we're going to apply a bunch of sinusoids to the same circuit and plot the frequency response, the ratio of the output versus input as a function of frequency.

And kept applying a variety of frequencies. So you can listen to the frequencies as they go by, and we will plot the amplitude up on the screen for you. Just for fun we are going to play frequencies between, say, 10 hertz and 20 kilohertz.

It will be fun for you to figure out at what point you stop hearing the frequencies. We are going to play from 10 hertz to 20 kilohertz. And figure out where your ears cut out. That will tell you what the break frequency of your ear is.

You can see the amplitude being articulated. The bottom figure is the phase. This is the frequency axis. This is the amplitude, log-log scales. I am not sure about you but I cannot hear anymore.

If you bring your canine friends to class it is quite possible that they would go berserk somewhere here. As I promised you, when I plot this on a log-log scale I get a straight line here and a straight line out there as well and the bottom line gives you the phase.

Now, what you can also do is you can also go to Websim. Websim is now linked on your course homepage. You can go to Websim and you can play with various L and C and R values. And if you plot frequency response, if you click on the frequency response button, boom, it will give you frequency responses for your circuit that look exactly like that.

You can go and play around with that. Thank you. All right. As the next step I promised to show you an easier path. And let's build some insight. Is there a simpler way to get where we would like to get? In particular, is there a simpler way to get Vp? Let's focus on Vp.

Why Vp? Because remember Vp was the complex amplitude of e to the j omega t. And once I know Vp then I know this expression here. Also notice that this here, the denominator is simply the characteristic equation for, I wonder how many of you noticed it, is simply the characteristic equation for the RC circuit.

If I can write down Vp, I can write down the characteristic equation, it will be in the denominator. I can also write down the frequency response very easily by taking the magnitude and phase of Vp.

So Vp has all the information humankind needs for those circuits. Is there a simpler way to get Vp? To bring some insight, let's go ahead and write down -- Let's stare at this for a while longer and see if light bulbs go off in our minds.

Of course, I could write this as Vi/(1+sRC). I just replaced the shorthand notation for a j omega. And I simply divide by SC throughout. So I get Vi times, I simply divide by SC throughout. Here is Vi.

I have one by SC, one by SC plus R. Light bulbs beginning to go off? The form we have here is 1/SC, some function of my capacitance divided by something connected to my capacitance plus R. This is Vi multiplied by something connected to capacitance divided by something connected to capacitance plus R.

And remember your circuit. What is that reminiscent of? What does that remind you of? Voltage divider? Hmm. There is some voltage divider thing going on here. I just cannot quite pin it. It is something about the capacitor, capacitor plus booster, some voltage divider thingamajig happening here.

We will try to figure that out. What I will do is replace those terms with something called Zc. Zc plus Zr. If I can find out the Zr and Zc somehow, I can write down the Vp by inspection by the voltage divider action, by some generalization of the good old Ohm's law that I know about.

Let's proceed further and see if we can make some kind of a connection between this and this. If I can make the connection then boom, I'm done. I will just use voltage dividers and I am home. OK, so let's play around and see.

There is something in there. By now you should know that we are very close. There is something going on in there. I just need to get that spark. I just need to make that spark so I can bridge the gap between something that is really easy versus where I am.

Let's take a look at the resistor. I have my resistor with the voltage vR across it and a current iR. Remember to get to any sort of steady state you are going to be dealing with the drives of the form vI e to the j omega t, exponential drives.

And by taking the real part, I know I get the input, and the real part of the output gives me the actual output. Let's say my iR is simply Ire^st and my vR is Vre^st. The S is, again, a shorthand notation for j omega.

If my current Ire^st of the exponential form shown there and here is Vr, I need to find out what relates Vr and Ir for the element relationship for the resistor to hold. In general, Ir and Vr are complex numbers.

For the resistor, I know that Vr=RIr. And I substitute using my complex drives here. So it is Vre^st=RIre^st. I am just substituting for these drives, Ohm's law should apply, and I cancel off e^st.

And so I get Vr=RIr. Interesting. For the resistor I find that, based on the fundamental principles of resistor action, the complex amplitude of the voltage simply relates to the complex amplitude of the input by the proportionality factor R.

In other words, for the resistor -- Just as the time domain V and I were related by the proportionality constant R, the complex amplitudes Vr and Ir are also related in the same way. That's interesting.

Now let's look at the capacitor. Some current ic flowing through it and a voltage vc. Let's say the current is Ice^st and the voltage is Vce^st. Let's plug these into the element law for the capacitor and see if we can find out a way of relating vc and ic.

I know that ic is simply Cdvc/dt. So I replace this with Ice^st=Cd/dt(vce^st), which is simply Ice^st=CsVce^st. So I can cancel this out again. Interesting. Ic=CsVc. Very interesting. What is interesting here? Notice that in the time domain Ic=Cdvc/dt, the element law for the capacitor.

So I said let's use exponential drives, Ice^st, Vce^st, that's an exponential drive, and try to find out what the relationship between the complex amplitudes are. I plug them and what do I find? I find that if my input is Vce^st, and Vc is the amplitude of the input, then the current is simply given by something multiplied Vc.

It's very similar in form to what I saw here. The resistor, Vr=RIr. For the capacitor, Vc=Ic/sc. 1/sc kind of plays the role of R. In other words, the complex amplitudes around the capacitor are related by Vc equals some constant times Ic.

Almost like a funny Ohm's law kind of relationship where Vc and IC are complex amplitudes. For the inductor it is the same way, iL, vL and L. Let's say iL=Ile^st and vL=Vle^st. Substitute the values for the inductor into its element relationship as well.

I know that vL=LdiL/dt. Therefore, substituting the complex amplitudes is L. And diL/dt will simply be Ilse^st. So I cancel out the exponentials. The reason we're able to do all of this is simply the remarkable beauty of exponentials.

Exponentials are absolutely stunningly beautiful. The reason is that when I differentiate them what I get back is the exponential times some constant, and the constant was in its numerator multiplying t.

And that's the beauty of exponentials. If this was a sine then I would get cosine and a sine. With exponentials these cancel out and what I am left with is something that is LsIl. Again, for the inductor, the voltage across the inductor relates to some constant Ls here times Il.

This is absolutely stunning and almost looks like a form of Ohm's law here. What I am going to do is let's give this the name Zr. Let's give this 1/sC the name Zc. And let's give this the name ZL.

It kind of behaves like a resistor, so the resistor simply becomes Zr. And 1/sC behaved like a resistor so I called it Zc. And this is a ZL. These are called "impedances". In other words, for a capacitor, as far as complex inputs and outputs are concerned, if Vc and Ic is fed to it, the capacitor can be replaced by an impedance Zc where I can write the relationship between Vc and Ic as Vc=ZcIc.

Where Zc is simply one by sc. Similarly, for an inductor -- -- I can write its impedance ZL as sL and I get Vl=ZLIl. And finally for a resistor it is pretty simple. What I am saying is that if I am in the region of the playground, if I constrain myself in the region of the playground where my inputs are something Vi e to the j omega t or exponentials, in that little region of the playground now, I am focusing more and more on small parts of the playground so I am kind of boxed in right now.

In that region of the playground this applies. In that region of the playground, I can replace resistors by impedances, capacitors with impedances of value 1/sC. And within that playground the beauty of analysis there is that in that region of the playground where the inputs are of the form Vie^st, it turns out that the element laws are simply generalizations of Ohm's law.

That is absolutely stunning. It is one of the biggest hallelujah moments in learning circuits. This is really big. And I think this is almost as big as the realization that you can take a nonlinear circuit, operate it at a given operating point, and you can sit around doing Zen things, looking at small perturbations in there, those are going to be linearly related.

This is one of the big hallelujah moments in 6.002. And this is of the same magnitude as the small signal response being linear. It is something that is completely non-intuitive. It is something that you just would not have known until you had seen it happen.

The same way here. This is very important so I will repeat it again. I have boxed myself into this small region of the playground where all I care about are sinusoidal inputs and steady-state responses.

So there I focus on complex inputs, Vi e to the j omega t. And I have just shown you that I can replace inductors, capacitors, resistors with their impedances. And the amplitudes of the corresponding signals around them are related by just a simple Ohm's law like relationship using impedances.

I am sort of boxed into this playground, right? In my playground it is all about e to the ij omega t. e to the ij omega t is implicit everywhere. I just don't show it. If I want to talk to somebody else outside but within MIT in this small region, it's all e to the ij omega t in there.

If I want to talk to somebody outside, get out of MIT, get out of this playground, what else do I have to do? I have to take the real part. Don't forget that. Remember that, take for example Vc here, so Vc is this, so implicit in all of this is that if I measure Vc at some place it is really going to be Vce to the j omega t.

And if we the cosine, the real part, then I have to take a real part of this. And the real part of that would Vc cosine of omega t angle Vc. This piece here kind of goes unsaid. We will agree that we have to do it, but we just skip that step because it is obvious.

We just deal with Vcs and Vls now. So a new notation certainly sneaked by you, and that notation looks like a big letter and a small letter. Remember you have seen vL, this is the total behavior, you have seen vl, that's a small signal behavior, and now you see this, Vl, capital V small l.

And we also have DC, we have labeled operating point values as VL, capital V, capital L. We have one thing left so nobody go out there inventing something new because we would be in trouble. This is capital V, small l, and this is simply "complex amplitude" in the small boxed region of my playground where good things happen and exponentials fly.

Whenever someone gives you a variable, capital V, small l, remember it's a complex amplitude, a complex number, and you know how to get to the time domain from there. You take that number, take the real part, multiple the number by e to the j omega t and take the real part, which is tantamount to magnitude cosine omega t plus angle of that number.

Actually, you know what? Let's send this up. Back to an example. Oh, I'm sorry. I'm sorry. This is not good. This is my time domain circuit. Remember this was my time domain circuit. A vI input.

A vC output. I wanted to analyze this. What I am telling you now is let's box ourselves in this impedance playground. And in the impedance playground the input becomes the complex amplitude of the input, my resistance gets replaced by a box Zr, my capacitor gets replaced by a box Zc.

And the voltage I care about here is Vc. Zr = R and Zc=1/sC. Now, there we go. I can write down Vc using a voltage divider action as Vc is simply Zc/(Zc+Zr), done, times Vi of course. And that gives me 1/sC divided by 1/sC+R and multiplying throughout by sC I get 1/1+sCR where S is j omega.

Just cannot get any simpler. How long did I take to do this? 30 seconds. Where I spent a whole lecture on Tuesday grinding through first trig, giving up halfway and collapsing, and then showing you the sneaky path which was still pretty painful, but 30 seconds, boom.

This stuff is spectacularly beautiful. The really cool thing here is that in this impedance domain for linear circuits all your good old tricks apply. Your Thevenin, your Norton, your superposition, name it and it applies for this linear circuit.

If you close your eyes and make believe that Zr is like an R and simply apply all the techniques you have learned so far in this linear playground. Just a little hack at the end where this is the complex amplitude.

And if you want to go to the time domain part then you do the usual thing. Modulus Vc cosine omega t plus angle Vc. Just remember that. That's the jump to get back to the time domain. Just to show you that this not just works for one little rinky-dink circuit here, let me take a more complicated circuit.

If I believe in my own BS, I should be able to apply this theory to my series RLC, the big painful circuit that we did differential equations for about a week ago. Let's do it. I have an inductor, a capacitor and a resistor.

What I am going to do is replace this with the impedance model. Input Vi. Let's say this was vI. Let's say I cared about vR. L, C and R. The impedance model would simply be Vi. What's the impedance of an inductor? SL.

And for the capacitor it is 1/sC. And for a resistor it is simply R. And just remember, if I can find out VR then for an input cosine of the form Vi cosine omega t the output will given by |Vr| cosine of omega t plus angle Vr.

Just remember this last step. But Vr itself is trivially determined. It is the voltage divider action again times Vi. And the voltage divider action is in the denominator I sum these thingamajigs, so ZL+ZC+ZR, ZR in the numerator.

And Zr is simply R. ZL is sL. Zc is 1/sC. And R is R. Vi. And I multiply through by, in this particular situation, by s/L. I want to get it into the same form as you've seen before. Multiply throughout, the numerator and denominator by s/L, what do I get? I get RS/L and out here I end up getting S squared plus 1/LC, and I get plus R/L S.

I am done. Look at that. Well, a little more than 30 seconds. Maybe a minute. What is this? Where have you seen this before? The denominator of this expression here? Ah, characteristic equation for the RLC.

Remember I promised you in the beginning that when we come to the end of the day using a simple one-minute expression I am going to write down the characteristic equation? Boom, here is what I get. Did somebody hear an echo in there? Notice that just by doing a simple voltage divider thingamajig, I got this expression.

And now I can write down the frequency response by replacing s is equal to j omega. Even more beautiful and what is even more stunningly pretty here is that remember the intuitive method I taught you about? The characteristic equation gives you alpha, omega nought, omega d and Q.

And based on those we can sketch even the time domain response. Guess what? RLC circuits are passé now. You can just write this thing down and you're done, 30 seconds or less. No DEs, no trig, no nothing.

OK.