3.60 Symmetry, Structure and Tensor Properties of Materials

SOME SOLUTIONS TO THIRD-ORDER EQUATIONS

BASIC EQUATION: $y^3 + py^2 + qy + r = 0$

TO CONVERT TO "NORMAL FORM" LET $y = x - p/3$

THEN $x^3 + ax + b = 0$

WITH $\begin{cases} a \equiv \frac{1}{3}(3q - p^2) \\ b \equiv \frac{1}{27}(2p^3 - 9pq + 27r) \end{cases}$

SOLUTIONS:

$\boxed{x_1 = A + B}$ $\quad$ $\boxed{x_2, x_3 = -\frac{1}{2}(A+B) \pm i\frac{\sqrt{3}}{2}(A-B)}$

WHERE:

$$A = \left\{-\frac{b}{2} + \left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{\frac{1}{2}}\right\}^{1/3}$$

$$B = \left\{-\frac{b}{2} - \left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{\frac{1}{2}}\right\}^{1/3}$$

IF p, q, r ARE REAL, THEN IF:

$\frac{b^2}{4} + \frac{a^3}{27} > 0$ THERE ARE 1 REAL, 2 CONJUGATE IMAGINARY ROOTS

$\frac{b^2}{4} + \frac{a^3}{27} = 0$ " " 3 REAL ROOTS, TWO OF WHICH ARE EQUAL

$\frac{b^2}{4} + \frac{a^3}{27} < 0$ " " 3 REAL, UNEQUAL ROOTS

UNFORTUNATELY, THE THIRD CASE IS THE ONE MOST FREQUENTLY ENCOUNTERED IN PRESENT APPLICATIONS. IN THIS INSTANCE THE ABOVE RELATIONS ARE OF LITTLE USE (NOTE THE TERM $\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{\frac{1}{2}}$ IN THE DEFINITION OF THE PARAMETERS A AND B ABOVE)

THE THREE FOLLOWING ALTERNATIVE EXPRESSIONS FOR THE ROOTS MAY THEN BE USEFUL:

IF $\frac{b^2}{4} + \frac{a^3}{27} < 0$:

$$\boxed{x_K = -2\frac{b}{|b|}\left(-\frac{a}{3}\right)^{\frac{1}{2}} \cos\left(\frac{\phi}{3} + 120^\circ \cdot K\right) \qquad K = 0, 1, 2}$$

WITH $\cos\phi = \left[\frac{b^2}{4} \cdot \left(-\frac{27}{a^3}\right)\right]^{1/2}$

IF $\frac{b^2}{4} + \frac{a^3}{27} = 0$

$$\boxed{\text{ROOTS ARE: } -2\frac{b}{|b|}\left(-\frac{a}{3}\right)^{\frac{1}{2}}, \quad \frac{b}{|b|}\left(-\frac{a}{3}\right)^{\frac{1}{2}} \text{ TWICE}}$$

IF $\frac{b^2}{4} + \frac{a^3}{27} > 0$

$$\boxed{\text{THE REAL ROOT IS } x = 2\frac{b}{|b|}\left(\frac{a}{3}\right)^{\frac{1}{2}} \operatorname{CTN} 2\phi}$$

WITH $\tan\phi = (\tan\psi)^{\frac{1}{3}}$

AND $\operatorname{CTN} 2\psi = \left(\frac{b^2}{4} \cdot \frac{27}{a^3}\right)^{\frac{1}{2}}$