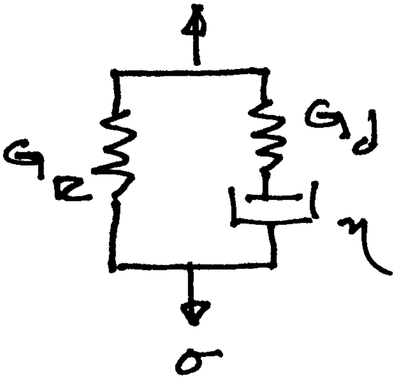
Zener Model



$$\gamma = \gamma_R = \gamma_d$$

$$\sigma = \sigma_R + \sigma_d \quad , \quad \sigma_R = G_R \gamma$$

$$\dot{\gamma} = \dot{\gamma}_d + \dot{\gamma}_\eta = \frac{1}{G_d}\dot{\sigma}_d + \frac{1}{\eta}\sigma_d$$

$\mathcal{L}, \times G_d$:
$$G_d \cdot s\bar{\gamma} = s\bar{\sigma}_d + \frac{G_d}{\eta}\bar{\sigma}_d = \left(s + \frac{1}{\tau_\sigma}\right)\bar{\sigma}_d$$

$$\bar{\sigma}_d = \frac{G_d \cdot s\bar{\gamma}}{s + \frac{1}{\tau_\sigma}}$$

$$\bar{\sigma} = \bar{\sigma}_R + \bar{\sigma}_d = G_R\bar{\gamma} + \frac{G_d \cdot s\bar{\gamma}}{s + \frac{1}{\tau_\sigma}}$$

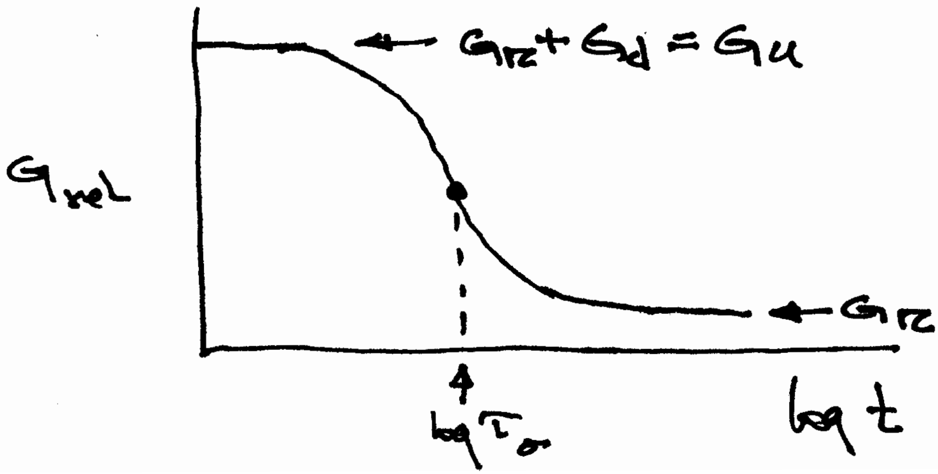
$$= \left(G_R + \frac{G_d s}{s + \frac{1}{\tau_\sigma}}\right)\bar{\gamma} \qquad (\bar{\sigma} = \mathcal{G}\bar{\gamma})$$

Relaxation: $\gamma(t) = \gamma_0 u(t) \rightarrow \bar{\gamma} = \gamma_0 / s$

$$\bar{\sigma} = \left(G_R + \frac{G_d s}{s + \frac{1}{\tau_\sigma}} \right) \cdot \frac{\gamma_0}{s}$$

$$\frac{\bar{\sigma}_0}{\gamma_0} = \bar{G}_{rel} = \frac{G_R}{s} + \frac{G_d}{s + \frac{1}{\tau_\sigma}}$$

$$G_{rel}(t) = G_R + G_d e^{-t/\tau_\sigma}$$



Constant strain rate:

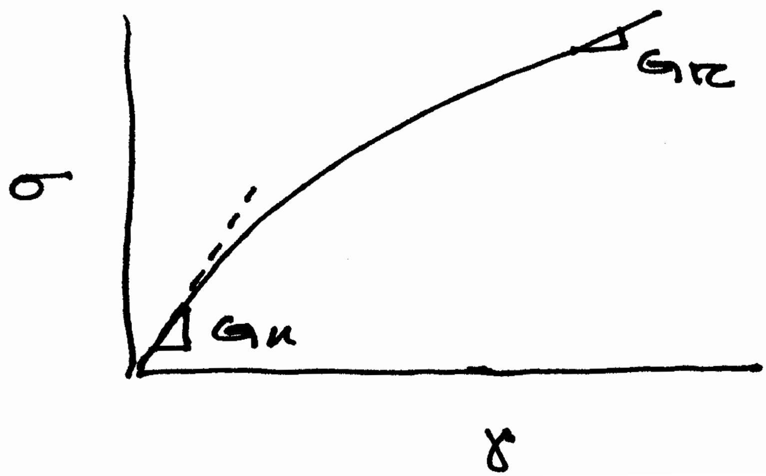
$$\gamma(t) = R_\gamma \cdot t \rightarrow \bar{\gamma} = R_\gamma / s^2$$

$$\bar{\sigma} = \left(G_R + \frac{G_d\, s}{s + \frac{1}{\tau_\sigma}} \right) \cdot \frac{R_\gamma}{s^2} = R_\gamma \left(\frac{G_R}{s^2} + \frac{G_d}{s(s+\frac{1}{\tau_\sigma})} \right)$$

$$\sigma(t) = G_R \cdot R_\gamma t + G_d R_\gamma \tau_\sigma \left(1 - e^{-t/\tau_\sigma} \right)$$

$$\frac{d\sigma}{d\epsilon} = \frac{d\sigma}{dt} \cdot \frac{dt}{d\gamma} = G_R + G_d e^{-t/\tau_\sigma} \equiv G_{rel}(t)$$

$$\frac{1}{R_\gamma}$$



Dynamic loading (DMA)

Laplace-plane shear operator

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> G[L]:=G[R]+ (G[d]*s)/(s+1/tau[sigma]);
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$$G_L := G_R + \frac{G_d\, s}{s + \frac{1}{\tau_\sigma}}$$

Applied strain in time plane:

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> unprotect(gamma);gamma(t):=gamma[0]*cos(omega*t);
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$$\gamma(t) := \gamma_0 \cos(\omega\, t)$$

Applied strain in laplace plane:

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> with(inttrans):gamma(s):=laplace(gamma(t),t,s);
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$$\gamma(s) := \frac{\gamma_0\, s}{s^2 + \omega^2}$$

Dynamic modulus in laplace plane:

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> G_bar:=G[L]*gamma(s)/gamma[0];
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$$G\_bar := \frac{\left(G_R + \frac{G_d\, s}{s + \frac{1}{\tau_\sigma}}\right) s}{s^2 + \omega^2}$$

Invert for time-plane modulus:

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> G_t:=invlaplace(G_bar,s,t);
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$$G\_t := \frac{G_d\, \mathbf{e}^{\left(-\frac{t}{\tau_\sigma}\right)}}{\omega^2\, {\tau_\sigma}^2 + 1} - \frac{\omega\, \tau_\sigma\, G_d \sin(\omega\, t)}{\omega^2\, {\tau_\sigma}^2 + 1} + \frac{G_R\, \omega^2\, {\tau_\sigma}^2 \cos(\omega\, t)}{\omega^2\, {\tau_\sigma}^2 + 1} + \frac{G_R \cos(\omega\, t)}{\omega^2\, {\tau_\sigma}^2 + 1} + \frac{\omega^2\, {\tau_\sigma}^2\, G_d \cos(\omega\, t)}{\omega^2\, {\tau_\sigma}^2 + 1}$$

Simplifying:

```
> 'G(t)'=factor(collect((G_t),cos(omega(t))));
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$$\mathrm{G}(t) = \frac{G_d\, \mathbf{e}^{\left(-\frac{t}{\tau_\sigma}\right)} - \omega\, \tau_\sigma\, G_d \sin(\omega\, t) + G_R\, \omega^2\, {\tau_\sigma}^2 \cos(\omega\, t) + G_R \cos(\omega\, t) + \omega^2\, {\tau_\sigma}^2\, G_d \cos(\omega\, t)}{\omega^2\, {\tau_\sigma}^2 + 1}$$

Simplifying further and rearranging manually:

$$G^* = \frac{G_d}{1+\omega^2\tau_\sigma^2} e^{\frac{-t}{\tau_\sigma}} + \left(G_R + \frac{G_d\, \omega^2 \tau_\sigma^2}{1+\omega^2\tau_\sigma^2}\right)\cos(\omega t) - \left(\frac{G_d\, \omega \tau_\sigma}{1+\omega^2\ {}_\sigma^2}\right)$$