

22.01 Fall 2016, Quiz 1 Solutions

October 26, 2016

Quiz Instructions: Answers can be given symbolically or graphically, no calculation is necessary. *No calculators, devices, or anything else allowed, except one double-sided, 8.5 x 11 inch sheet of paper.* Define any intermediate variables which you need to complete the problems. Generous partial credit will be given for showing correct methodology, even if the solution is not given.

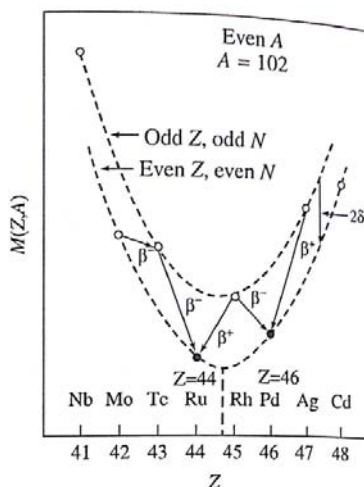
1 (48 points) Short Answers, 6 points each

Each of these problems can be solved with one sentence, one equation, and/or one graph/picture.

1.1 Which types of isotopes can decay by both positron (β^+) and beta (β^-) decay? Be specific about their nuclear structure.

Answer: *Odd-Z, odd-N*

Explanation: Typically even-Z and/or even-N nuclei are more stable, as nuclear shell levels are filled in pairs. The origin of the instability of odd-Z/odd-N nuclei comes from having two unpaired nucleons in the highest energy level. Using the semi-empirical mass formula, one can see that the binding energy is lower for odd-Z/odd-N nuclei, because the pairing term is negative. This results in more stable nuclei on either side as a function of Z, or in the mass parabola sense, it represents the case where there are two mass parabolas for a given mass number A, like the figure below.



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1.2 Write a necessary and sufficient inequality that describes when a nuclear reaction is energetically allowed to occur.

Answer: $Q + T_i \geq 0$

Explanation: If the Q value is positive (exothermic), then the reaction can always occur. If the Q value is negative (endothermic), then extra energy must be supplied in the form of kinetic energy of the incoming particle.

1.3 What is the least likely emitted anti-neutrino energy during beta decay?

Answer: 0

Explanation: The diagram below shows the energy spectrum of beta particles (left) and antineutrinos (right). One can see that the least likely beta particle energy to observe is $E_{\beta^-} = Q$, therefore the least likely antineutrino energy to observe is $E_{\bar{\nu}} = Q - E_{\beta^-} = 0$

1.4 Which nuclear decay type(s) could directly or indirectly cause Auger electron emission to occur, and why?

Answer: *Positron decay* (β^+), *electron capture* (*EC*), *internal conversion* (*IC*), *spontaneous fission* (*SF*)

Explanation: Any process that causes electron emission will leave orbital electron holes, and Auger electron emission is a competing process to characteristic x-ray emission. Positron decay causes the emission of an orbital electron to conserve charge. Electron capture directly grabs an orbital electron. Internal conversion ejects an electron as the gamma ray gets out. Spontaneous fission leaves the fission products in an ionized state, and the neutralization of this charge consists of electrons falling down in orbital shells.

1.5 Why can't we round atomic masses when performing energetics calculations? Give an example where a rounding error would be severe.

Answer: $1 \text{ amu} = 931.49 \text{ MeV}$; Beta decay (rounding to 931MeV would remove a whole electron rest mass)

Explanation: Even rounding to 931.5 MeV would introduce an error of 10keV to a binding energy or excess mass calculation, which can make it impossible to determine true energy level transitions and radioactive decay energies.

1.6 What is the kinetic energy of a nucleus following alpha decay?

Answer: $E_{\alpha} = Q \frac{M_{\text{recoil}}}{M_{\alpha} + M_{\text{recoil}}}$

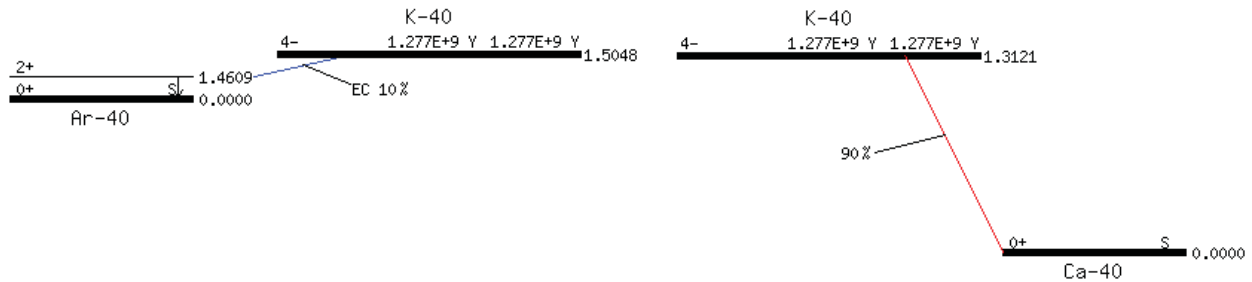
Explanation: By conservation of momentum and energy, an alpha particle can never carry away the full nuclear reaction energy Q . The recoil nucleus must carry away equal momentum as the alpha particle.

1.7 Why must uncertainties be added in quadrature, not as a simple sum? Write the formula for the combined uncertainty of two experiments together, each with their own uncertainty.

Answer: Summing uncertainties of random variables overestimates the total error. $\sigma_{\text{net}} = \sqrt{\sigma_{\text{gross}}^2 + \sigma_{\text{background}}^2}$

Explanation: Simply summing errors does not account for the fact that truly random errors can partially cancel each other. They have equal probability of being of the same or the opposite sign. Summing them in quadrature is like calculating the magnitude of a vector in error space.

1.8 Suppose you measure 10^5 gamma ray counts from a ^{40}K source in 100,000 seconds, using a 0.01% efficient detector. What is the activity of your ^{40}K source in Becquerels (Bq), and what is the uncertainty of the activity? Ignore background counts. Use the following diagram to help you:



Answer: $10^5 \pm 1 \text{ Bq}$

Explanation: 10^5 counts in 10^5 seconds is a measured count rate of 1 CPS. A detector with 0.01% efficiency means that a measured rate of 1 CPS means an actual count rate of 10,000 CPS. We know that only 10% of the counts come from gamma transitions, while the other 90% come from beta decay, which we don't count. Therefore, a measured count rate of 1 CPS means an actual disintegration rate of 100,000 CPS. The uncertainty is calculated as $\sigma = \sqrt{c/t}$, and the *true* count rate is 100,000 CPS. That leads to an uncertainty of $\sqrt{100,000/100,000}$ CPS. We also ignore any solid angle effects in this solution, assuming that both the intrinsic detector efficiency and geometrical efficiency (the solid angle over 4π) are accounted for in the 0.01% efficiency.

2 (26 points) Breeding Reactor Fuel

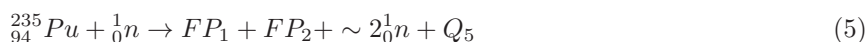
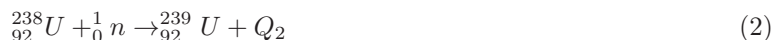
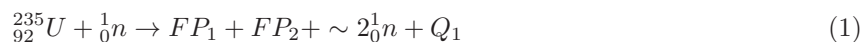
Reminder: Answers can be given symbolically or graphically, no calculation is necessary.

Reactors can breed their own fuel, by converting ^{238}U , a fertile element, to ^{239}Pu , a fissile element like ^{235}U . ^{238}U can capture a neutron, which then decays via beta decay to ^{239}Np , which decays via beta decay to ^{239}Pu . Key data for this problem include:

$$\sigma_{f,U-235} = 500b \quad \sigma_{f,Pu-239} = 500b \quad \sigma_{c,U-238} = 5b$$

$$t_{1/2,U-239} = 1.5 \cdot 10^3 \text{ sec} \quad t_{1/2,U-238} = 2 \cdot 10^5 \text{ sec} \quad \Phi = 10^{14} \frac{n}{\text{cm}^2\text{s}}$$

2.1 (5 points) Write the complete nuclear reactions or radioactive decay reactions for each of the following isotopes: ^{235}U , ^{238}U , ^{239}U , ^{239}Np , ^{239}Pu .



2.2 (10 points) Draw a graph of the total amount of fissile material in the reactor versus time. You may want to start by writing differential equations describing the production (by neutron capture) and destruction (radioactive decay and/or fission) of each isotope.

First, let's look at the relative half lives and reaction rates, to see if we can forget about any.

Rxn	Type	Expression	Value
1	Artificial	$\Phi\sigma_{f,U-235}$	$5 \cdot 10^{-8}\text{s}^{-1}$
2	Artificial	$\Phi\sigma_{c,U-238}$	$5 \cdot 10^{-10}\text{s}^{-1}$
3	Natural	λ_{U-239}	$5 \cdot 10^{-4}\text{s}^{-1}$
4	Natural	λ_{Np-239}	$5 \cdot 10^{-6}\text{s}^{-1}$
5	Artificial	$\Phi\sigma_{f,Pu-239}$	$5 \cdot 10^{-8}\text{s}^{-1}$

We have used the relation $\lambda = 0.693/t_{1/2}$, and because it's an exam, we just say that $0.693 \approx 1$ to make the calculations easier. We can see that reactions 3 and 4 are incredibly quick compared to the others, they are 100x and 10,000x faster than either fission reaction, respectively. Therefore, let's just pretend they don't exist for simplicity.

This yields the following differential equations:

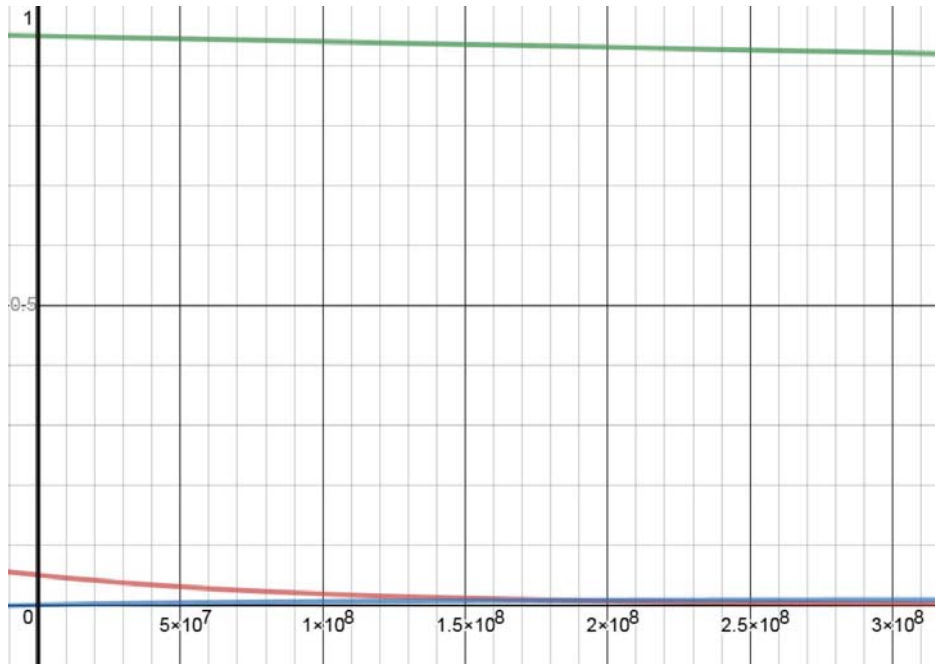
$$\frac{dN_{235U}}{dt} = -\Phi\sigma_{f,U-235}N_{235U}; \quad N_{235U}(t=0) = N_{235_0} \quad (6)$$

$$\frac{dN_{238U}}{dt} = -\Phi\sigma_{c,U-238}N_{238U}; \quad N_{238U}(t=0) = N_{238_0} \quad (7)$$

$$\frac{dN_{239Pu}}{dt} = \Phi\sigma_{c,U-238}N_{238U} - \Phi\sigma_{f,Pu-239}N_{239Pu}; \quad N_{239Pu}(t=0) = 0 \quad (8)$$

In Equation 6, we recognize that there is no source of ^{235}U , and it is destroyed by fission. In Equation 7, the exact same equation is found for ^{238}U , except this time the reaction is neutron capture, not fission. In Equation 8, every atom of ^{238}U makes an atom of ^{239}Pu , indirectly, but so quickly that we assume that the destruction of ^{239}U and the creation/destruction of ^{239}Np is relatively instantaneous.

Without solving anything, we can see that Equations 6 and 7 are just exponential decay functions, of the form $\frac{dN_I}{dt} = -\lambda_I N_I$, while Equation 8 is a standard production/destruction equation in the form of $\frac{dN_5}{dt} = \lambda_2 N_2 - \lambda_5 N_5$. We also note that $\lambda_2 \ll \lambda_5$, which means that the rate of ^{239}Pu fission is much faster than the rate of ^{239}Pu creation. Therefore, we don't expect much to build up at all. We therefore get a graph which looks like this:



Here we have assumed that the amount of ^{235}U in the reactor at $(t = 0)$ is 5%, while the amount of ^{238}U is 95%. You can use whatever starting values you like, or keep it symbolic. Note that the amount of N_I (^{235}U , red) decays over time, just like if it were radioactive decay. N_2 (^{238}U , green) also decays exponentially, but at a far lower rate because of the far lower capture cross section. Meanwhile, N_5 (^{239}Pu , blue) builds up, and decays almost as rapidly as it's made. The slope of N_5 is directly proportional to the value of N_2 at $(t = 0)$, which follows directly from Equation 8. For full credit, you should recognize the following:

1. Both ^{235}U and ^{238}U undergo exponential decay, but ^{238}U decays much more slowly.
2. ^{239}Pu barely builds up at all, because its fission rate is 100x faster than its creation rate.
3. The level of ^{239}Pu should remain very low, and constant after the initial build-up period, because ^{238}U is burned so slowly that the rate of production of ^{239}Pu remains almost constant over time.

NOTE: The following is not required for credit, but we will show it for completeness.

The solutions to these equations are quite simple, straight from the book/lectures/homework. Equations 6 and 7 are simple exponential decays:

$$N_{235U} = N_{2350} e^{-\Phi\sigma_{f,U-235}t} \quad (9)$$

$$N_{238U} = N_{2380} e^{-\Phi\sigma_{c,U-238}t} \quad (10)$$

Equation 8 takes the general form of $N_2(t) = \frac{\lambda_2 N_{10}}{(\lambda_2 - \lambda_1)} [e^{-\lambda_2 t} - e^{-\lambda_1 t}]$ (see Problem Set 4 solutions for a worked example of how to find this solution):

$$N_{239Pu} = \frac{\Phi\sigma_{c,U-238}N_{10}}{(\Phi\sigma_{f,Pu-239} - \Phi\sigma_{c,U-238})} [e^{-\Phi\sigma_{c,U-238}t} - e^{-\Phi\sigma_{f,Pu-239}t}] \quad (11)$$

2.3 (11 points) Write an expression for the breeding ratio, or the rate of fissile material creation vs. fissile material burning.

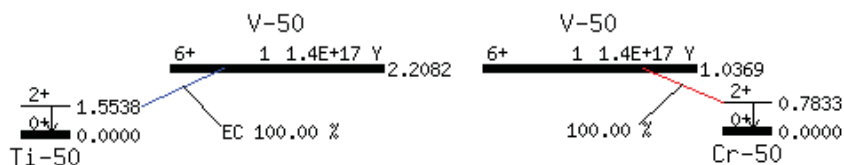
This is actually simpler than it may look. The rate of fissile material creation over the rate of fissile material destruction is just the production and destruction terms from Equations 6 and 8, with the positive ones (production) on top and the negative ones (destruction) on the bottom:

$$Ratio = \frac{\cancel{\phi}\sigma_{c,U-238}N_{238U}}{\cancel{\phi}\sigma_{f,U-235}N_{235U} + \cancel{\phi}\sigma_{f,Pu-239}N_{239Pu}} = \frac{\left(\beta \cdot 10^{-16} \text{s}^{-1}\right) N_{238U}}{\left(\beta \cdot 10^{-8} \text{s}^{-1}\right) N_{235U} + \left(\beta \cdot 10^{-8} \text{s}^{-1}\right) N_{239Pu}} = \frac{N_{238U}}{100(N_{235U} + N_{239Pu})} \quad (12)$$

Full credit will be given for the symbolic answer.

3 (26 points) Decay Chain Diagrams

For these problems, consider the decay of ^{50}V , which decays by multiple paths as shown below:



3.1 (8 points) Write the complete nuclear reactions for all possible decays shown, assuming that ^{50}Ti and ^{50}Cr are stable isotopes.

$$^{50}\text{V} \rightarrow ^{50}\text{Ti}^* + \nu_e + 0.6544 \text{ MeV} \quad (EC, \text{ no } \beta^+ \text{ possible}) \quad (13)$$

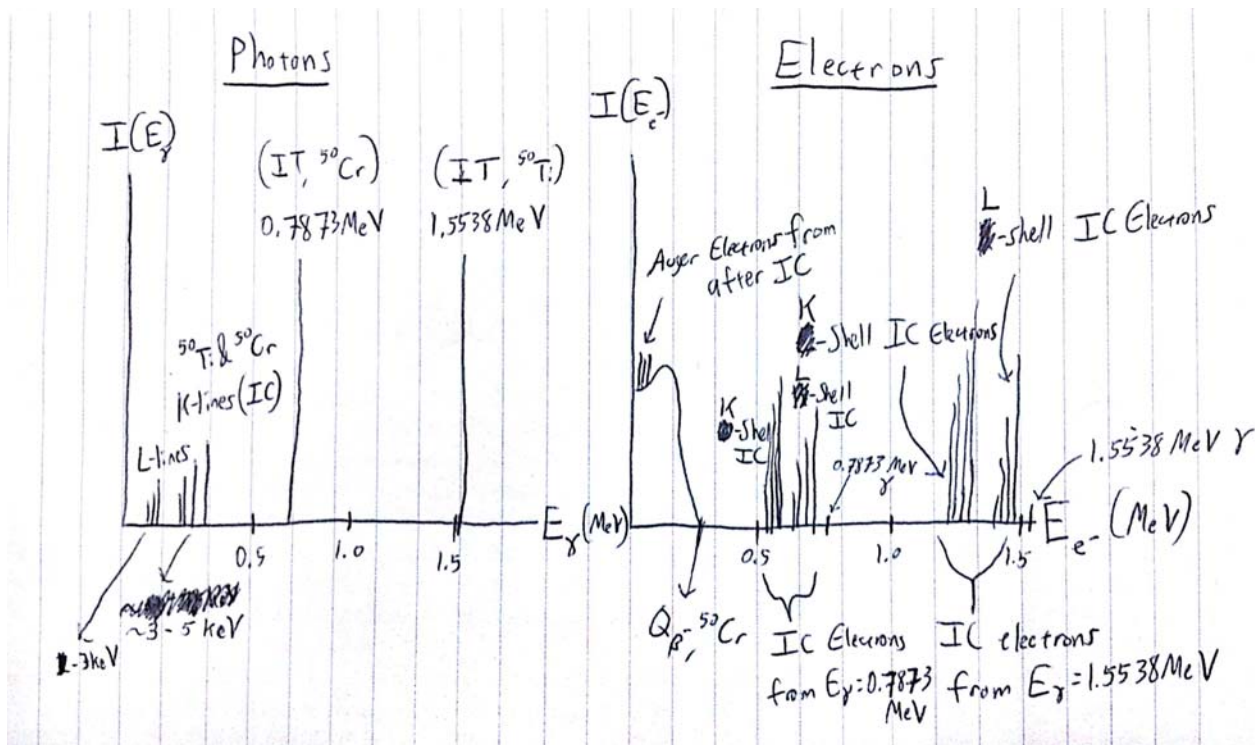
$$^{50}\text{Ti}^* \rightarrow ^{50}\text{Ti} + \gamma \quad (E_\gamma = 1.5538 \text{ MeV}) \quad (IT, \text{ may also undergo IC, } x\text{-ray emission, Auger emission}) \quad (14)$$

$$^{50}\text{V} \rightarrow ^{50}\text{Cr}^* + \beta^- + \bar{\nu}_e + 0.2536 \text{ MeV} \quad (\beta^- \text{ decay}) \quad (15)$$

$$^{50}\text{Cr}^* \rightarrow ^{50}\text{Cr} + \gamma \quad (E_\gamma = 0.7833 \text{ MeV}) \quad (IT, \text{ may also undergo IC, } x\text{-ray emission, Auger emission}) \quad (16)$$

3.2 (10 points) Draw complete photon and electron spectra which would be observed from the decay of ^{50}V .

A fully complete answer, worth full credit, is given as follows:



The relative intensity of things doesn't matter much, except to know that L-shell transitions are less likely than K-shell ones. One point each will be given for putting the following features in the correct energy locations:

1. Photon Spectrum:

- Gamma (γ) transition from ^{50}V to ^{50}Ti at 1.5538 MeV
- Gamma (γ) transition from ^{50}V to ^{50}Cr at 0.7873 MeV
- K-shell x-rays produced by electrons from internal conversion (IC) filling lower level shells, at around 4-7 keV. Ti and Cr are so atomically close in Z that their K-lines are hard to distinguish.
- L-shell x-rays produced by electrons from internal conversion (IC) filling lower level shells, at around 1-3 keV, a *lower* energy than photons originating in the K-shell due to their lower binding energy. Ti and Cr are so atomically close in Z that their L-lines are hard to distinguish.

2. Electron Spectrum:

- Beta particle (β^-) decay spectrum, ranging from 0 MeV to the maximum of 0.2536 MeV, with a peak at roughly 1/3rd of the Q value.
- K-shell internal conversion (IC) electrons at about 4-7 keV below 1.5538 MeV, ejected when a 1.5538 MeV gamma ray hits a K-shell electron on its way out of the $^{50}\text{Ti}^*$ nucleus.
- L-shell internal conversion (IC) electrons at about 1-3 keV below 1.5538 MeV, ejected when a 1.5538 MeV gamma ray hits a L-shell electron on its way out of the $^{50}\text{Ti}^*$ nucleus, at a *higher* energy than the K-shell electrons due to their lower binding energy.
- K-shell internal conversion (IC) electrons at about 4-7 keV below 0.7873 MeV, ejected when a 0.7873 MeV gamma ray hits a K-shell electron on its way out of the $^{50}\text{Cr}^*$ nucleus.
- L-shell internal conversion (IC) electrons at about 1-3 keV below 0.7873 MeV, ejected when a 0.7873 MeV gamma ray hits a L-shell electron on its way out of the $^{50}\text{Cr}^*$ nucleus, at a *higher* energy than the K-shell electrons due to their lower binding energy.
- Auger electrons emitted anywhere in the range of a few eV to a few hundred eV, as a competing process to internal conversion (IC) x-ray emission following electron ejection.

3.3 (6 points) Why do nuclei like ^{50}V decay by either mechanism shown here, while most isotopes only have one mode of decay?

^{50}V is an odd-odd isotope (odd-N, odd-Z), so both its outermost proton and neutron shell levels are partially filled. Therefore, it can increase its binding energy per nucleon by either gaining or losing a proton, like the double mass parabola example in Problem 1.1. Or, using the semi-empirical mass formula, there is a negative pairing term for odd-odd nuclei, lowering their binding energy and making them less stable.

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