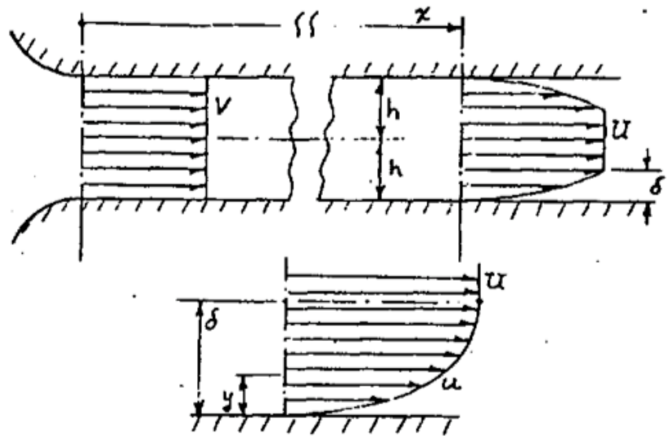
MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Problem 9.09

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



Consider the two-dimensional, incompressible, steady flow of a fluid of constant viscosity in the entrance region of a slit of width $2h$. The flow enters the tube ($x = 0$) with the uniform velocity V. At distance x from the entrance, the boundary layer thickness is δ, and the core flow has the speed U. The boundary layer is laminar.

The ultimate objective is to analyze the entrance region, using momentum integral method, to determine how the boundary layer thickness, the skin friction stress, and the pressure gradient all vary with distance x. Since it is known that the fully-developed flow has parabolic velocity profile, it is agreed to assume, with some approximation, that the boundary layer velocity profile in the entrance region is also parabolic, following the equation

$$\frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 \qquad (9.09a)$$

Carry your analysis only to the point of obtaining a differential equation relating the dimensionless boundary layer thickness δ/h as dependent variable to the dimensionless length x/h as the independent variable. Any constants of the problem may of course appear in the differential equation. Do not attempt to integrate the latter.

Solution:

Velocity profile in boundary layer assumed parabolic and it is give as

$$\frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 \qquad in \qquad 0 \le y \le \delta \tag{9.09b}$$
$$u = U \qquad in \qquad \delta \le y \le h \tag{9.09c}$$

Let's apply mass conservation to determine $U = U(x)$. At region (1) and (2), the volume flow rates are respectively

$$\int_0^h V\,dy = Vh \qquad at \qquad (1) \tag{9.09d}$$
$$\int_0^h u\,dy = \int_0^\delta 2U\frac{y}{\delta} - U\left(\frac{y}{\delta}\right)^2\,dy + \int_\delta^h U\,dy \tag{9.09e}$$
$$= \left[\frac{U}{\delta}y^2 - \frac{U}{3\delta^3}y^3\right]_0^\delta + U(h-\delta) \tag{9.09f}$$
$$= \cdots \quad = U\left(h - \frac{1}{3}\delta\right) \qquad at \qquad (2) \tag{9.09g}$$

$$\Rightarrow V = U\left(1 - \frac{1}{3h}\delta\right) \Rightarrow \boxed{U(x) = \frac{V}{1 - \frac{\delta(x)}{3h}}} \tag{9.09h}$$

Now let's consider Karman momentum integral equation (See Kundu textbook p.362 - 364 for derivation).

$$\boxed{\frac{d}{dx}\left(U^2\theta\right) + \delta^* U\frac{dU}{dx} = \frac{\tau_o}{\rho}} \tag{9.09i}$$

where the displacement thickness δ^* and momentum thickness θ are defined as

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right)\,dy\,, \qquad \theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right)\,dy \tag{9.09j}$$

Plugging these into the integral equation gives

$$\Rightarrow \underbrace{\frac{d}{dx}U^2\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)\,dy}_{A} + \underbrace{U\frac{dU}{dx}\int_0^\delta\left(1-\frac{u}{U}\right)\,dy}_{B} = \underbrace{\frac{\tau_o}{\rho}}_{C} \tag{9.09k}$$

where the integral range has been replaced from 0 to "δ" because $1 - u/U$ is zero outside boundary layer. Let's calculate each term. If we substitute the given velocity profile into "A" is, then it is

$$\boxed{A}\ : \qquad \frac{d}{dx}U^2 \int_0^{\delta} \left(2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 \right) \left(1 - 2\frac{y}{\delta} + \left(\frac{y}{\delta}\right)^2 \right) \ dy \tag{9.09l}$$

For simplicity, let's define

$$\eta(x) \equiv \frac{y}{\delta(x)} \Rightarrow dy = \delta d\eta \tag{9.09m}$$

then

$$\Rightarrow \quad \frac{d}{dx}\left(U^2\delta\right) \int_0^1 \left(2\eta - \eta^2\right)\left(1 - 2\eta + \eta^2\right) \ d\eta \tag{9.09n}$$

$$= \quad \cdots = \underline{\frac{2}{15}\frac{d}{dx}\left(U^2\delta\right)} \tag{9.09o}$$

Now let's calculate the second term $\boxed{B}$. Using η again,

$$\boxed{B}\ : \qquad U\frac{dU}{dx}\delta \int_0^1 \left(1 - 2\eta + \eta^2\right) \ d\eta = \cdots = \underline{\frac{1}{3}\delta U \frac{dU}{dx}} \tag{9.09p}$$

Finally, $\boxed{C}$ becomes

$$\boxed{C}\ : \qquad \frac{\tau_o}{\rho} = \nu \left(\frac{\partial u}{\partial y}\right)_o = \nu U \left(\frac{2}{\delta} - \frac{2y}{\delta^2}\right)_{y=0} = \underline{2\frac{\nu U}{\delta}} \tag{9.09q}$$

Hence, gathering $\boxed{A}$, $\boxed{B}$ and $\boxed{C}$ yields

$$\boxed{9\delta U \frac{dU}{dx} + 2U^2\frac{d\delta}{dx} = 30\frac{\nu U}{\delta}} \tag{9.09r}$$

We cannot solve this equation because both U and δ are function of x but we have only one equation. For the relation between U and δ, we can use Equation (9.09g) which was obtained by mass conservation. For this, we may use the following algebras.

$$\frac{dU}{dx} = -\frac{V}{\left(1 - \frac{\delta(x)}{3h}\right)^2} \cdot \left(-\frac{1}{3h}\right) \cdot \frac{d\delta}{dx} \tag{9.09s}$$

$$U\frac{dU}{dx} = \frac{V^2}{\left(1 - \frac{\delta(x)^3}{3h}\right)^2} \cdot \left(\frac{1}{3h}\right) \cdot \frac{d\delta}{dx} \tag{9.09t}$$

Substitute these into the integral equation gives

$$9\delta \frac{V^2}{\left(1-\frac{\delta(x)^3}{3h}\right)^2} \cdot \left(\frac{1}{3h}\right) \cdot \frac{d\delta}{dx} + 2\frac{V^2}{\left(1-\frac{\delta(x)}{3h}\right)^2}\frac{d\delta}{dx} = 30\frac{\nu}{\delta}\frac{V}{\left(1-\frac{\delta(x)}{3h}\right)} \tag{9.09u}$$

Therefore, finally we get

$$\Rightarrow V\frac{d\delta}{dx}\left(\frac{1}{1-\frac{1}{3}\frac{\delta}{h}}\right)\left(\frac{3\frac{\delta}{h}}{1-\frac{1}{3}\frac{\delta}{h}}+2\right) = 30\frac{\nu}{\delta} \tag{9.09v}$$

Using dimensionless boundary layer thickness $\bar{\delta}$ and length $\bar{x}$, which are defined as

$$\underline{\bar{\delta} \equiv \frac{\delta}{h}, \qquad \bar{x} \equiv \frac{x}{h}} \tag{9.09w}$$

the differential equation can be simplified as

$$\boxed{Vh\left(\frac{2+7/3\bar{\delta}}{\left(1-1/3\bar{\delta}\right)^2}\right)\bar{\delta}\frac{d\bar{\delta}}{dx} = 30\nu} \tag{9.09x}$$

This is an ODE in terms of dimensionless parameters, which shows that $\bar{\delta}$ is a function of $\bar{x}$, and by Equation (9.09g) U is also a function of $\bar{x}$ as well as pressure gradient $\rho U dU/dx$. In addition, we can see τ_o varies with $x(\bar{x})$.

Note that $\rho U \frac{dU}{dx}$ is pressure gradient in boundary layer because the order-of-magnitude of the pressure gradient term in the x momentum equation is estimated from Euler's equation applied to the outer inviscid flow at the edge of the boundary layer. Since $v \sim O(\delta)$, $v \ll u$ and Euler's equations reduces to

$$U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \sim O(1) \tag{9.09y}$$

□

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