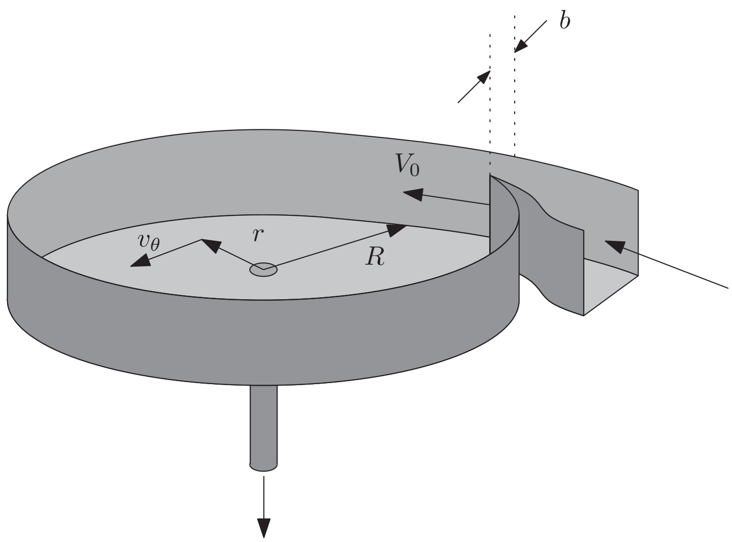
MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Problem 5.32

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



The steady sink flow in the sketch is set up by injecting water tangentially through a narrow channel near the periphery and letting it drain through a hole at the center. The vessel has a radius R. At the point of injection, the water has a velocity V and depth h_0; the width of the injection channel, b, is small compared with R.

In what follows, we consider the region of the flow act too close to the drain, and assume that everywhere in the region (i) the flow is essentially incompressible and inviscid, (ii) the radial velocity component $|v_r|$ is small compared with the circumferential velocity component v_θ, and at the periphery.

(a) Show, by applying the angular momentum theorem to a control volume comprising the water between $r = r$ and $r = R$, that

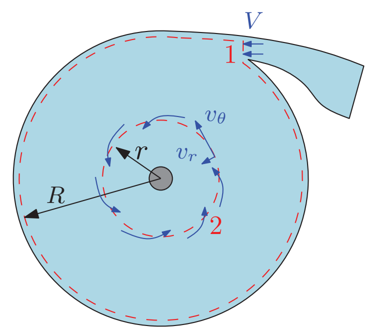
$$v_\theta = VR/r$$

(b) Show that the assumption $|v_r| \ll v_\theta$ is satisfied if $b \ll R$.

Solution:

Assume: incompressible, inviscid flow, $v_r \leq v_\theta$

CS	$\mathbf{v}$	$\hat{\mathbf{n}}$	$\mathbf{v}_c$	$(\mathbf{v}-\mathbf{v}_c)\cdot\hat{\mathbf{n}}$
1	$V\hat{\mathbf{e}}_\theta$	$-\hat{\mathbf{e}}_\theta$	0	$-V$
2	$v_\theta\hat{\mathbf{e}}_\theta - v_r\hat{\mathbf{e}}_r$	$-\hat{\mathbf{e}}_r$	0	v_r



(a) Angular Momentum Conservation:

$$\cancel{\sum \mathbf{T}} = \underbrace{\cancel{\frac{d}{dt}\int_{CV}[\mathbf{r}\times\rho\mathbf{v}]\,dV}}_{\text{steady}} + \int_{CS}[\mathbf{r}\times\rho\mathbf{v}](\mathbf{v}-\mathbf{v}_c)\cdot\hat{\mathbf{n}}\,dA \tag{5.32a}$$

$$CS(1): \qquad \mathbf{r}\times\mathbf{v} = \begin{vmatrix}\hat{\mathbf{e}}_r & \hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ R & 0 & 0 \\ 0 & V & 0\end{vmatrix} = RV\hat{\mathbf{e}}_z$$

$$CS(2): \qquad \mathbf{r}\times\mathbf{v} = \begin{vmatrix}\hat{\mathbf{e}}_r & \hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ r & 0 & 0 \\ -v_r & v_\theta & 0\end{vmatrix} = rv_\theta\hat{\mathbf{e}}_z$$

Therefore, Eq. (5.32a) becomes

$$-\rho RV \underbrace{\int_{CS,(1)} V\,dA}_{Q_{\text{in}}} + \rho r v_\theta \underbrace{\int_{CS,(2)} v_r\,dA}_{Q_{\text{out}}} = 0$$

By Mass Conservation, $Q_{\text{in}} = Q_{\text{out}}$ $\qquad \therefore \qquad RV = rv_\theta$

$$\Rightarrow \boxed{v_\theta = \frac{RV}{r}} \tag{5.32b}$$

(b) Going back to mass conservation,

$$Q_{\text{in}} = Vbh_0 = v_r \cdot 2\pi r \cdot h(r) = Q_{\text{out}}$$

Substitute in $V = rv_\theta/R$:

$$\frac{v_\theta b h_0 \cancel{r}}{R} = v_r 2\pi \cancel{r} h(r)$$

Assuming $2\pi h(r)$ and h_0 are the same order of magnitude,

$$v_r \approx v_\theta\left(\frac{b}{R}\right)$$

Finally, if $\frac{b}{R} \ll 1$,

$$\boxed{v_r \ll v_\theta} \tag{5.32c}$$

* Recall 4.21 and how much HARDER it was to solve using Bernoulli !!

□

Problem Solution by Sungyon Lee, Fall 2005

