**Problem 6.05**
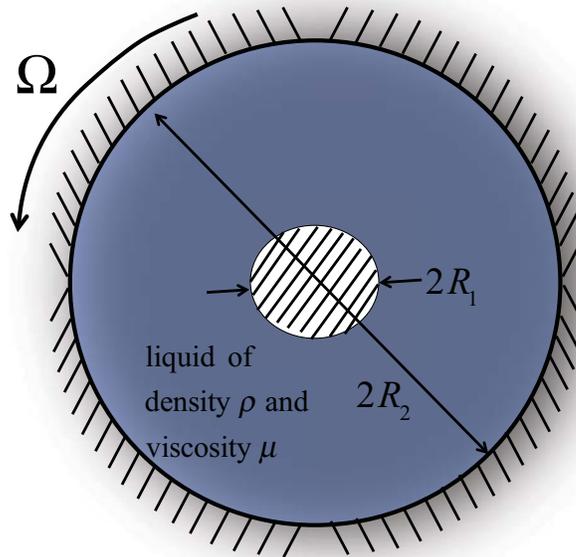*This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin*



Figure 1: Geometry of the problem.

The general definition of the coefficient of viscosity, as applied to two-dimensional motions, is

$$-\mu \equiv \frac{\tau}{d\gamma/dt} \tag{6.05a}$$

where $\tau$ is the shear stress and $d\gamma/dt$ is the rate of change of the angle between two fluid lines which at time $t$ are mutually perpendicular, the rate of change being measured by an observer sitting on the center of mass of the fluid particle.

- (a) Show that in terms of streamline coordinates,

$$\tau = \mu \left( dV/dn - V/R \right) \tag{6.05b}$$

  where $V$ is the resultant velocity, $R$ is the radius of curvature of the streamline, and $n$ is the outward-going normal to the streamline.

- (b) A long, stationary tube of radius $R_1$ is located concentrically inside of a hollow tube of inside radius $R_2$, and the latter is rotated at constant angular speed $\omega$. The annulus contains fluid of viscosity $\mu$. Assuming laminar flow, and neglecting end effects, demonstrate that

$$\frac{P}{\mu \omega^2 R_2^2} = \frac{4\pi}{(R2/R1)^2 - 1} \tag{6.05c}$$

  where $P$ is the power required to turn unit length of the hollow tube.

- (c) Find the special form of (b) as $R_2/R_1 \to 1$, in terms of the gap width $h = R_2 - R_1$ and the radius R.

**Solution:**

- (a) Consider two fluid lines perpendicular to each other at time $t$ for a particle at position $P$ and select one of the lines to be parallel to the velocity vector/streamline $ds$. The other line will be consequently parallel to $dn$. Now track the particle till it reaches point $P'$ at time $t + dt$. Figure 2 shows the mentioned geometry and depicts the possible deformations as the fluid particle travels on a streamline. from the definition we have the following:
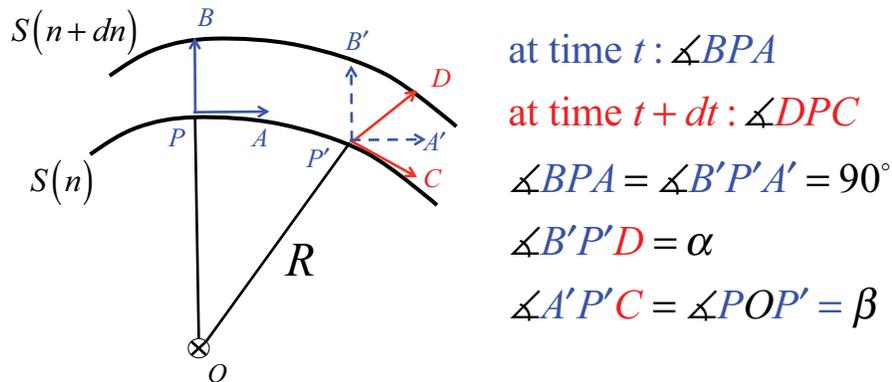


$$\text{at time } t : \sphericalangle BPA$$
$$\text{at time } t + dt : \sphericalangle DPC$$
$$\sphericalangle BPA = \sphericalangle B'P'A' = 90°$$
$$\sphericalangle B'P'D = \alpha$$
$$\sphericalangle A'P'C = \sphericalangle POP' = \beta$$

Figure 2: Deformation for a particle traveling from $P$ at time $t$ to $P'$ at time $t + dt$ along a streamline.

$$-\mu \equiv \frac{\tau}{d\gamma/dt} \tag{6.05d}$$

As shown in Figure 2 it is easy to see that $-d\gamma = \alpha - \beta$ so one can write:

$$\mu \equiv \frac{\tau}{\alpha/dt - \beta/dt} \tag{6.05e}$$

On the other hand we have the following relationship for $\alpha/dt$:

$$\alpha/dt = (B'D/dn)/dt = (BD - B'B)/dndt = (BD - PP')/dndt = (V(n+dn)dt - V(n)dt)/dndt \tag{6.05f}$$

which follows to this:

$$\alpha/dt = \frac{dV}{dn} \tag{6.05g}$$

for $\beta/dt$ one can see that:

$$\beta/dt = \sphericalangle POP'/dt = PP'/Rdt = Vdt/Rdt \tag{6.05h}$$

so we will have:

$$\beta/dt = \frac{V}{R} \tag{6.05i}$$

from (e), (g), and (i) it is easy to see that:

$$\tau = \mu\left(dV/dn - V/R\right) \tag{6.05j}$$

Note: if you feel that the geometry relationships are hard to visualize try to solve this problem assuming that your coordinate system is locally cylindrical with center $O$ and your motion (locally) has only $V_s$ which is $V_\theta$ so $V_r = 0$. Using the relationships for strain rate in cylindrical coordinates (you can find them in Kundu) you will get something exactly similar to the geometry proof.

- (b) Consider the $\theta$-component in the Navier Stokes equations for cylindrical coordinates:

$$\frac{\partial V_\theta}{\partial t} + (\underline{V}.\underline{\nabla}) V_\theta + \frac{V_r V_\theta}{r} = \frac{-1}{\rho} \frac{1}{r} \frac{\partial P}{\partial \theta} + \nu \left( \nabla^2 V_\theta + \frac{2}{r^2} \frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r^2} \right) \qquad (6.05\text{k})$$

in which:

$$(\underline{V}.\underline{\nabla}) = V_r \frac{\partial}{\partial r} + \frac{V_\theta}{r} \frac{\partial}{\partial \theta} + V_z \frac{\partial}{\partial z} \qquad (6.05\text{l})$$

and

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \qquad (6.05\text{m})$$

Now if you consider that $\partial/\partial\theta = 0$ due to axisymmetry in the problem and also note that $V_r = V_z = 0$ then (k) simplifies to

$$0 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_\theta}{\partial r} \right) - \frac{V_\theta}{r^2} \qquad (6.05\text{n})$$

thus:

$$\boxed{r^2 \frac{\partial^2 V_\theta}{\partial r^2} + r \frac{\partial V_\theta}{\partial r} - V_\theta = 0} \qquad (6.05\text{o})$$

Note that (o) can be rewritten as:

$$\frac{d}{dr} \left( r^2 \frac{dV_\theta}{dr} - r V_\theta \right) = 0 \Rightarrow \left( r^2 \frac{dV_\theta}{dr} - r V_\theta \right) = const. \qquad (6.05\text{p})$$

if we divide (p) by $r^3$ we will then get:

$$\frac{1}{r} \frac{dV_\theta}{dr} - \frac{1}{r^2} V_\theta = const/r^3. \Rightarrow \frac{d}{dr} \left( \frac{V_\theta}{r} \right) = \frac{const}{r^3}. \qquad (6.05\text{q})$$

thus:

$$\frac{V_\theta}{r} = \frac{c_1}{r^2} + c_2 \Rightarrow \boxed{V_\theta = \frac{c_1}{r} + c_2 r}. \qquad (6.05\text{r})$$

in which $c_1$ and $c_2$ are two integration constants which need to be determined from boundary conditions. Boundary conditions for this steady problem are:

$$\left\{ \begin{array}{l} at \ r = R_1 : \ \ V_\theta = 0 \\[2em] at \ r = R_2 : \ \ V_\theta = R_2 \Omega \end{array} \right\} \qquad (6.05\text{s})$$

Satisfying the boundary conditions one will get the following values for $c_1$ and $c_2$:

$$\left\{ \begin{array}{l} c_1 = -\frac{R_1^2 R_2^2 \Omega}{R_2^2 - R_1^2} \\[2em] c_2 = \frac{R_2^2 \Omega}{R_2^2 - R_1^2} \end{array} \right\} \qquad (6.05\text{t})$$

which leads to the following velocity distribution:

$$\boxed{\frac{V_\theta}{R_2 \Omega} = \frac{R_2 r}{R_2^2 - R_1^2} - \frac{R_2 R_1^2}{\left( R_2^2 - R_1^2 \right) r}} \qquad (6.05\text{u})$$

Note that at the limit of $R_1 \to 0$ the velocity field becomes exactly similar to solid body rotation i.e., $V_\theta = r\Omega$.

Now that we have the velocity distribution we can easily calculate $\tau_{r\theta}$:

$$\tau = \mu \left( dV/dn - V/R \right) = \mu \left( dV/dr - V/r \right) = \frac{2\mu R_1^2 R_2^2 \Omega}{\left( R_2^2 - R_1^2 \right) r^2} \qquad (6.05\text{v})$$

thus at $r = R_2$:

$$\boxed{\tau = \frac{2\mu R_1^2 \Omega}{(R_2^2 - R_1^2)}}$$

(6.05w)

For calculating the required power per unit length $(P)$ at $r = R_2$ we have:

$$P = F_{shear}U_{wall} = 2\pi R_2 \tau_{wall} R_2 \Omega$$

(6.05x)

Plugging the result from (w) in (x) we will have:

$$\boxed{\frac{P}{\mu\omega^2 R_2^2} = \frac{4\pi}{(R2/R1)^2 - 1}}$$

(6.05y)

- (c) Now in the limit where $R_2/R_1 \to 1$ we have $R_2 = R_1 + h \Rightarrow R_2/R_1 = 1 + h/R$ in which $R = R_1$. If we plug this in the relationship for stress (v) we can easily show that at the limit of $R_2/R_1 \to 1 \Rightarrow R_1 \approx R_2 \approx R$ we will have::

$$\tau = \frac{2\mu R_2^2 \Omega}{\left[(1 + h/R)^2 - 1\right] r^2} \approx \frac{R_1}{2h}\frac{2\mu R_2^2 \Omega}{r^2} \approx \frac{\mu R\Omega}{h}$$

(6.05z)

Notice that (z) shows that at the limit of $R_2/R_1 \to 1$ the shear stress is almost constant in the gap and its values is very close to what we had in simple plane Couette flow. This fact plus the benefits of circular and long geometries are the main ideas for making rheometers out of similar geometries (Taylor-Couette geometries). Plugging the value from (z) in the relationship for $P$ leads to $P/\left(\mu\omega^2 R^2\right) = 2\pi R/h$

$\square$

*Problem Solution by Bavand, Fall 2012*

2.25 Advanced Fluid Mechanics
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