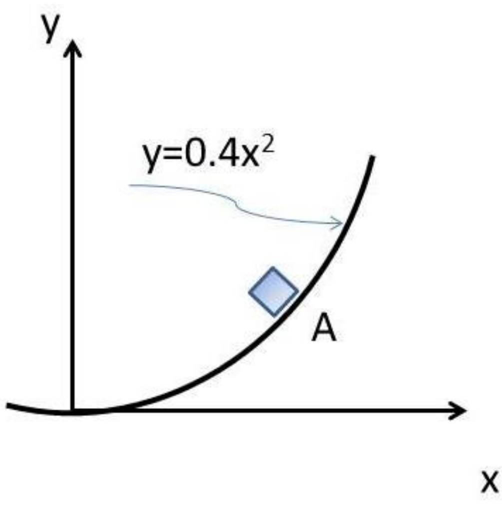
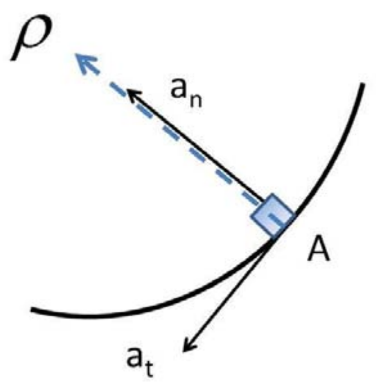
2.003 Engineering Dynamics
Problem Set 2 Solutions

This problem set is primarily meant to give the student practice in describing motion. This is the subject of kinematics. It is strongly recommended that you study the reading on kinematics provided with the course materials. You should also read the handout on velocities and accelerations of moving bodies, which is referred to in this solution.

Problem 1: A small box of negligible size slides down a curved path defined by the parabola $y = 0.4x^2$. When it is at point $A(x_A=2m, y_A=1.6m)$, the speed of the box is $V_{box} = 8m/s$ and the tangential acceleration due to gravity is $dV_{box}/dt = 4m/s^2$.

a. Determine the normal component and the total magnitude of the acceleration of the box at this instant.
b. Draw a free body diagram and determine if the tangential acceleration is consistent with the forces acting on the body. In particular, can you determine the friction coefficient which is consistent with the data given?





a. The normal component of the acceleration is caused by the curvature of the path. The radius of curvature may be obtained from a well-known relationship from analytic geometry.

$y = 0.4x^2$

At (x,y)=(2m,1.6m)

$$\frac{dy}{dx} = 0.8x\,|_{x=2} = 0.8(2.0) = 1.6$$

$$\frac{d^2y}{dx^2}\,|_{x=2} = 0.8m^{-1}$$

The radius of curvature is given by:

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1+1.6^2\right]^{3/2}}{0.8} = 8.4m$$

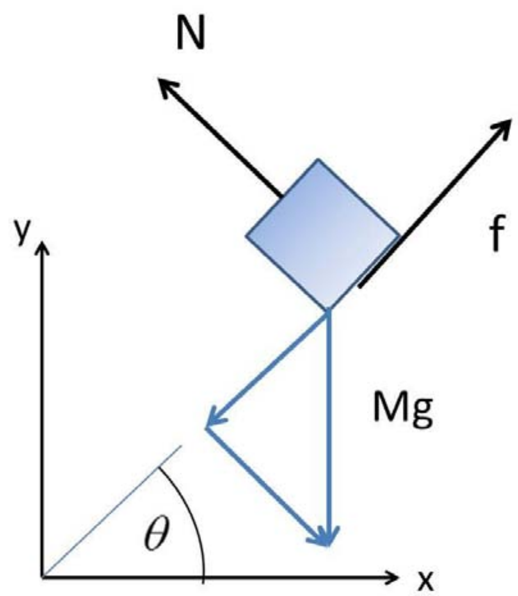
The total acceleration on the box is the sum of the normal and tangential components.

$$\vec{a}_{A/O} = a_t\hat{u}_t + a_n\hat{u}_n$$

$$a_t = 4m/s^2$$

$$a_n = \frac{v_{A/O}^2}{\rho} = \frac{(8m/s)^2}{8.4m} = 7.623m/s^2$$

$$\left|\vec{a}_{A/O}\right| = \sqrt{a_n^2 + a_t^2} = 8.608m/s^2$$



b. The free body diagram is on the left. In the direction tangent to the slope the mass times the acceleration must be equal to the sum of the friction force and the tangential component of the gravitational force on the box. This is worked out in the equations below.

In the direction normal to the curve:

$$Ma_n = M\frac{v_{A/O}^2}{\rho} = \sum F_n = N - Mg\cos(\theta)$$

$$\theta = \tan^{-1}(\frac{dy}{dx}) = \tan^{-1}(1.6) = 58^\circ$$

$$\Rightarrow (1) \qquad N = M\frac{v_{A/O}^2}{\rho} + Mg\cos(\theta)$$

Tangential to the curve Newton's 2nd law provides that

$$(2) \qquad \sum F_t = Ma_t = -\mu N + Mg\sin(\theta) = M(4m/s^2)$$

Substituting for N from (1) into (2) leads to

$$Ma_t = -\mu\left(M\frac{v_{A/O}^2}{\rho} + Mg\cos(\theta)\right) + Mg\sin(\theta) = M(4m/s^2)$$

M cancels out and the remaining quantities are known, allowing μ to be found.

μ=0.337.

Problem 2: The 0.5-lb ball is guided along a circular path, which lies in a vertical plane. The length of the arm OP is equal to $r_{P/O} = 2r_C\cos\theta$. If the arm has an angular velocity $\dot{\theta} = 0.4\text{rad/s}$ and an angular acceleration $\ddot{\theta} = 0.8\text{rad/s}^2$ at the instant $\theta = 30^\circ$, determine the force of the arm on the ball. Neglect friction and the size of the ball. Set $r_C = 0.4\text{ft}$.

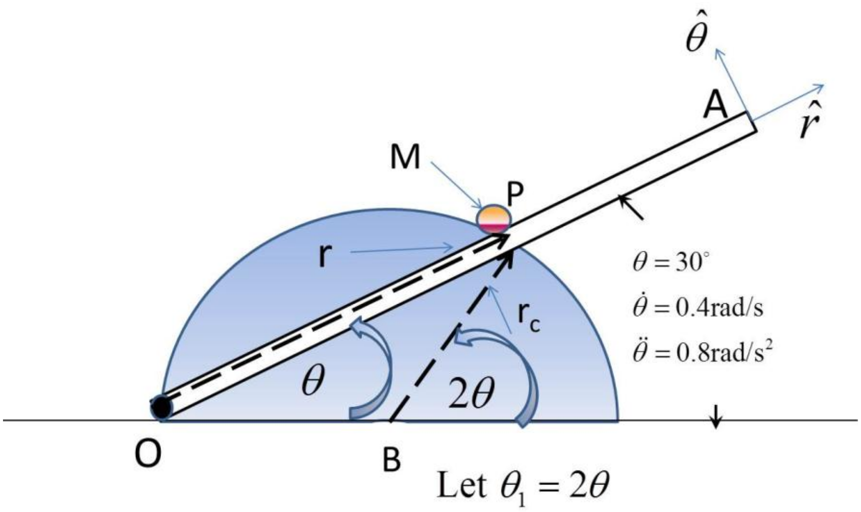


Figure 2

Solution: From the concept question we deduced that because friction is neglected, the force of the arm on the ball is perpendicular to the arm.

<u>We begin by describing the motion</u>: one of the most difficult steps in doing this problem is choosing a set of coordinates. There are many possibilities. In this case we seek

answers in terms of directions which are normal and parallel to the moving rod OA and the angular rotation rate and angular acceleration are specified for the rod. This suggests that an appropriate set of coordinates would be polar coordinates with origin at O, which describe the motion of the rod. The **r** vector is parallel to the rod, and the angle θ is defined as the angle between the rod and the horizontal, as shown in the figure.

The ball follows the semi-circular track. The two unknowns in the problem are the force F_n which is perpendicular to the curved track and the force F_P which is perpendicular to the rod. First the kinematics:

From trigonometry it is known that $r = 2r_c\cos(\theta)$. The other kinematic facts for the problem are:

$r_c = 0.4$feet

$\theta = 30$ degrees

$\dot{\theta} = 0.4$ radians/s

$\ddot{\theta} = 0.8$ radians/sec^2

$\vec{r} = r\hat{r}$

Expressions are needed for velocity and acceleration of the mass at point P. A set of polar coordinates, r and θ, are defined directly with respect to the inertial frame $O_{r\theta z}$:

$$\vec{V}_{P/O} = \left[\frac{d\vec{r}}{dt}\right]_{/O_{r\theta z}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \qquad (1)$$

$$\vec{a}_{P/O} = \left[\frac{d^2\vec{r}}{dt^2}\right]_{/O_{r\theta z}} = \left[\ddot{r} - r\dot{\theta}^2\right]\hat{r} + \left[r\ddot{\theta} + 2\dot{r}\dot{\theta}\right]\hat{\theta} \qquad (2)$$

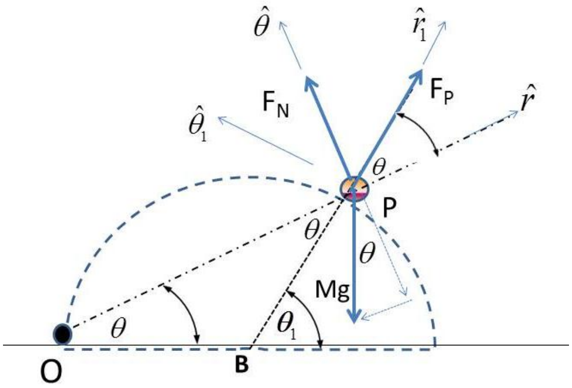
The set of unit vectors $\hat{r}$ and $\hat{\theta}$ define the direction of the **r** vector and the direction perpendicular to the **r** vector. $\hat{\theta}$ is perpendicular to $\hat{r}$, **but it is not a unit vector which defines the direction of the angle** θ. The angle θ is just a scalar coordinate which relates the position of the **r** vector in the reference frame in which it is defined. More on this confusing topic later on.

Expressions are needed for $\dot{r}$ and $\ddot{r}$, which may be obtained from $\vec{r} = 2r_c\cos(\theta)\hat{r}$. However these derivatives are tedious because of the time derivatives of the unit vectors that are required. It is in fact easier to compute the velocity and acceleration of the mass in terms of polar coordinates based on the radius r_c, which rotates about point B and use unit vectors $\hat{r}_1$ and θ_1. This is shown in the figure below. Once $\mathbf{V}_{P/O}$ and $\mathbf{a}_{P/O}$ are known in terms of the unit vectors $\hat{r}_1$ and $\hat{\theta}_1$ it is then possible to convert them into expressions in terms of the unit vectors $\hat{r}$ and $\hat{\theta}$. This is done using the following relations.

$$\begin{aligned}\hat{r}_1 &= \cos(\theta)\hat{r} + \sin(\theta)\hat{\theta} \\ \hat{\theta}_1 &= -\sin(\theta)\hat{r} + \cos(\theta)\hat{\theta}\end{aligned} \qquad (3)$$

This approach is easier because the mass exhibits pure circular motion about point B and the velocity and acceleration expressions are well known. Because point B is a fixed point in the

$O_{r\theta z}$ inertial frame, then a non-rotating frame at B is also an inertial frame. Velocities and accelerations measured with respect to a frame at B are the same as would be observed from the frame $O_{r\theta z}$. Therefore $\mathbf{V_{P/O}}$ and $\mathbf{a_{P/O}}$ measured with respect to B are the same as with respect to $O_{r\theta z}$.



Using polar coordinates with origin at point B, $\mathbf{V_{P/O}}$ and $\mathbf{a_{P/O}}$ are easily obtained.

$\vec{r}_1 = r_c \hat{r}_1$

Let $\theta_1 = 2\theta,\ \dot{\theta}_1 = 2\dot{\theta}$ and $\ddot{\theta}_1 = 2\ddot{\theta}$

$$\vec{V}_{P/O} = \left[\frac{d\vec{r}_1}{dt}\right]_{/O_{xyz}} = \dot{r}_1\hat{r}_1 + r_1\dot{\theta}_1\hat{\theta}_1 \qquad (4)$$

$$= 0 + r_c\dot{\theta}_1\hat{\theta}_1 = 2r_c\dot{\theta}\hat{\theta}_1$$

This is intuitively satisfying because it is simply circular motion about point B.

Taking another time derivative of $\vec{V}_{P/O}$ in equation (4) with respect to the fixed inertial frame leads to:

$$\vec{a}_{P/O} = \left[\ddot{r}_1 - r_1\dot{\theta}_1^{\,2}\right]\hat{r}_1 + \left[r_1\ddot{\theta}_1 + 2\dot{r}_1\dot{\theta}_1\right]\hat{\theta}_1$$

$$\vec{a}_{P/O} = \left[\ddot{r}_c - r_c\dot{\theta}_1^{\,2}\right]\hat{r}_1 + \left[r_c\ddot{\theta}_1 + 2\dot{r}_c\dot{\theta}_1\right]\hat{\theta}_1$$

$$\vec{a}_{P/O} = \left[0 - r_c\dot{\theta}_1^{\,2}\right]\hat{r}_1 + \left[r_c\ddot{\theta}_1 + 0\right]\hat{\theta}_1 = -2r_c\dot{\theta}^2\hat{r}_1 + 2r_c\ddot{\theta}\hat{\theta}_1$$

As expected of pure circular motion, there is a centripetal acceleration term and an Eulerian acceleration term. These can be expressed in terms of $\hat{r}$ and $\hat{\theta}$ unit vectors by the transformation given above in equation (3).

$$\hat{r}_1 = \cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}$$

$$\hat{\theta}_1 = -\sin(\theta)\hat{r} + \cos(\theta)\hat{\theta}$$

which leads to the following expression in terms of $\hat{r}$ and $\hat{\theta}$ unit vectors.

$$\vec{a}_{P/O} = -2r_c\dot{\theta}^2\left[\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}\right] + 2r_c\ddot{\theta}\left[-\sin(\theta)\hat{r} + \cos(\theta)\hat{\theta}\right] \qquad (5)$$

$$\vec{a}_{P/O} = \left[-2r_c\dot{\theta}^2\cos(\theta) - 2r_c\ddot{\theta}\sin(\theta)\right]\hat{r} + \left[-2r_c\dot{\theta}^2\sin(\theta) + 2r_c\ddot{\theta}\cos(\theta)\right]\hat{\theta} = -0.4309\frac{ft}{s^2}\hat{r} + 0.4903\frac{ft}{s^2}\hat{\theta} \qquad (6)$$

The above expression is not intuitively obvious. However it is now in terms of coordinates aligned with the rod OP, which makes it much easier to evaluate the normal force the rod exerts on the mass. This completes the kinematics portion of the problem.

Next the appropriate physical laws must be applied. In this case Newton's 2nd law is the primary one that is needed. By summing forces from the free body diagram two equations may be obtained relating force to acceleration of the mass in directions normal to and parallel to the rod. The problem has two unknown forces, $\mathbf{F_P}$ and $\mathbf{F_N}$, as shown in the figure above. $\mathbf{F_N}$ is the force that the rod exerts on the mass. It is perpendicular to the rod because there is no friction. $\mathbf{F_P}$ is the force on the mass exerted by the circular curve that the mass must follow. $\mathbf{F_P}$ is normal to the surface of the circular curve, again because there is no friction. The only other force acting on the mass is due to gravity.

All three forces must be reduced to components normal to and parallel to the rod. These are in the $\hat{r}$ and $\hat{\theta}$ directions. This leads to two equations in the two unknown forces.

$$\sum_{\hat{r}} F = -Mg\sin\theta + F_P\cos(\theta) = Ma_{r,P/O} \qquad (7)$$

$$\sum_{\hat{\theta}} F = -Mg\cos\theta + F_N + F_P\sin(\theta) = Ma_{\theta,P/O} \qquad (8)$$

where $a_{r,P/O}$ and $a_{\theta,P/O}$ are the $\hat{r}$ and $\hat{\theta}$ components of the acceleration $\vec{a}_{P/O}$

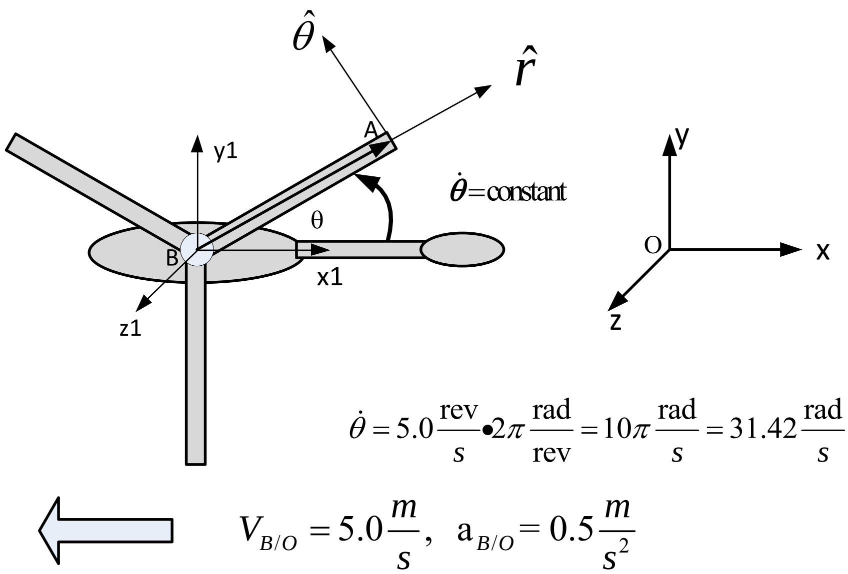
Equation (7) may be solved for F_P and substituted into equation (8). This final equation may be solved for F_N, the normal force of the rod on the mass. This is made dramatically simpler by first substituting in for all of the known angles, lengths, masses, velocities and accelerations in equations 5 through 8.

$$F_P = \left[Mg\sin(\theta) + Ma_{\hat{r},P/O}\right]/\cos(\theta) = 0.279\ lbs$$

$$F_N = -F_P\sin(\theta) + Mg\cos\theta + Ma_{\hat{\theta},P/O} = -0.1395 + 0.433 + [(0.0155)0.4263] = 0.30\ lbs$$

The forces are mostly due to the Mg term.

Problem 3 Solution – The blades of a helicopter rotor are of radius 5.2m and rotate at a constant angular velocity of 5rev/s. The helicopter is flying horizontally in a straight line at a speed of 5.0 m/s and an acceleration of 0.5m/s^2, as sketched in the figure. Determine the velocity and acceleration of point A, which is the outermost tip of one of the blades, (with respect to ground) when θ is 90^o.



$$\dot{\theta} = 5.0\frac{\text{rev}}{s}\bullet 2\pi\frac{\text{rad}}{\text{rev}} = 10\pi\frac{\text{rad}}{s} = 31.42\frac{\text{rad}}{s}$$

$$V_{B/O} = 5.0\frac{m}{s},\ \ \text{a}_{B/O} = 0.5\frac{m}{s^2}$$

$$\text{r}_{A/B} = 5.2m$$

Figure 3

The reference frames defined in the figure are as follows: O_{xyz} is fixed to ground; $B_{x1y1z1}$is fixed to the translating helicopter. This is a problem which asks only for a solution for velocity and acceleration. This a pure kinematics question. No physical laws need be applied. Just describe the motion and do the math.

In this solution we will begin with the full 3D vector equations expressing velocity and acceleration. Since this problem involves motion in only 2D, and the rotational aspect of the problem is simple circular motion, it may be simplified considerably, using polar coordinates, rather than using a rotating frame of reference, which is fixed to the rotating body.

In polar coordinates the **r** vector is allowed to rotate through an angle θ, which is defined with respect to a fixed reference frame, in this case B_{x1y1z1}. For simple circular motion, such as the blades of the helicopter, the **r** vector may be defined so as to rotate with the rigid body. In this problem we will require the **r** vector to rotate as if fixed to blade OA. From the notes provided on velocities and accelerations we have expressions in cylindrical coordinates, given below, for translating and rotating rigid bodies. Since this is motion confined to a plane the linear velocity and acceleration in the z direction are set to zero, resulting in formulas in simple polar coordinate form:

$$\vec{V}_{A/O} = \vec{V}_{B/O} + \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{k}$$

$$\vec{a}_{A/O} = \vec{a}_{B/O} + \ddot{z}\hat{k} + (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

For the helicopter problem many terms are zero: $\dot{r} = \ddot{r} = \dot{z} = \ddot{z} = \ddot{\theta} = 0,$ because the point A is fixed to the blade and there is no motion in the z direction. The angular acceleration is also given as zero. The angular velocity has to be converted to radians per second. Hence $\dot{\theta} = 2\pi(5 \text{ rev/s}) = 10\pi$ radians/second.

When all quantities are substituted in the velocity becomes:

$$\vec{V}_{A/O} = 5.0\frac{m}{s}\hat{i} + 5.2m(10\pi\frac{rad}{s})\hat{\theta} = 5.0\frac{m}{s}\hat{i} + 52\pi\frac{m}{s}\hat{\theta}$$

The answer is now in terms of unit vectors from two different coordinate systems. We need to convert to the unit vectors of the inertial reference frame O_{xyz} because that was what was specified in the problem. The unit vectors in the B_{x1y1z1} and the O_{xyz} frames align. Hence, $\hat{i} = \hat{i}_1,\ \hat{j} = \hat{j}_1,\ \hat{k} = \hat{k}_1$. The conversion of unit vectors from polar to Cartesian coordinates is given by:

$$\hat{r} = i_1\cos(\theta) + j_1\sin(\theta)$$
$$\hat{\theta} = -i_1\sin(\theta) + j_1\cos(\theta)$$

Substitution of these quantities leads to:

$$\vec{V}_{A/O} = -5.0\frac{m}{s}\hat{i} + 52\pi\frac{m}{s}\hat{\theta} = -5.0\frac{m}{s}\hat{i} - 52\pi\sin(\theta)\hat{i} + 52\pi\cos(\theta)\hat{j} = -(5 + 52\pi)\hat{i}$$
$$\vec{V}_{A/O} = -168.4\frac{m}{s}\hat{\mathrm{i}} \text{ at } \theta=\pi/2$$

The contribution to this velocity which comes from the forward velocity of the helicopter will increase with time due to the translational acceleration of the helicopter. This has been ignored in computing this answer.

The acceleration at A reduces to :

$$\vec{a}_{A/O} = \vec{a}_{B/O} + \ddot{z}\hat{k} + (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} = -0.5\frac{m}{s^2}\hat{i} - r\dot{\theta}^2\hat{r}$$

for $\theta=\pi/2$

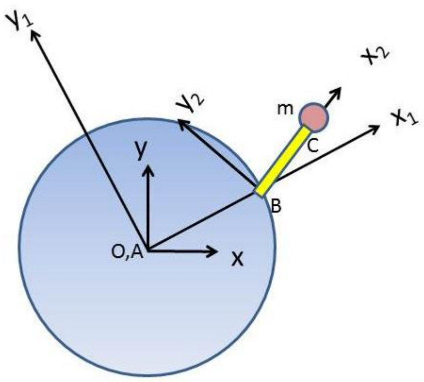
$$\vec{a}_{A/O} = -0.5\frac{m}{s^2}\hat{i} - 5.2m(10\pi \text{ rad/s})^2\hat{r} = [-0.5\hat{i} - 5132\hat{j}]\frac{m}{s^2}$$ The magnitude of the

acceleration is more **than 500g's**!! The forces on the rotor hub are very large.

A final note on polar coordinates: In this problem the coordinates $r, z,$ and θ are defined in the non-rotating B_{x1y1z1} frame which is attached to the body of the helicopter. They are not a rotating reference frame. In simple circular motion problems the rotating **r** vector, which may have components in the $\hat{r}$ and $\hat{k}$ directions, is adequate to describe the velocity and acceleration of a point on the rotating body. In such cases the need for an additional reference frame attached to the rotating body is avoided. It is necessary to use a rotating frame of reference attached to the body when velocities and accelerations must be described which cannot be accomplished with an **r** vector. An example would be to describe the velocity of a stone which has been struck by the helicopter blade. The **r** vector can account for the moving blade but not the velocity of the stone with respect to the blade. In such a case a rotating frame attached to the blade would be required.

Problem 4: An amusement park ride: A ride at an amusement park consists of a rotating horizontal platform with an arm connected at point B to the outside edge. The arm rotates at 10 rev/min with respect to the platform. The platform rotates at 6 rev/min with respect to the ground. The axes of rotation for both are in the z direction, out of the plane of the page as shown in the drawing. Gravity acts in the z direction. A seat is located at C at the end of arm BC, which is 2m long. The radius from A to B is 6 m long. The mass of the passenger is 75kg. Find the velocity and acceleration of the passenger.

The reference frames defined in Figure 4 are as follows: O_{xyz} is fixed to ground; A_{x1y1z1} is fixed to the platform at the axis of rotation. $B_{x2\ y2\ z2}$ is fixed to the arm BC at B and rotates with the arm.



Problem 4

Solution: For this solution we will practice using the full rotating coordinate system formulation.

Let"s begin by writing the general expression for the velocity of point C with respect to the inertial frame O_{xyz}, the vector sum of the velocities at B and the velocity of C with respect to B.

$$\vec{V}_{C/O}=\left(\frac{d\vec{r}_{C/O}}{dt}\right)_{/O_{xyz}}=\left(\frac{d\vec{r}_{B/O}}{dt}\right)_{/O_{xyz}}+\left(\frac{d\vec{r}_{C/B}}{dt}\right)_{/O_{xyz}}=\left(\frac{d(\vec{r}_{A/O}+\vec{r}_{B/A})}{dt}\right)_{/O_{xyz}}+\left(\frac{d\vec{r}_{C/B}}{dt}\right)_{/O_{xyz}}$$

but, $\vec{r}_{A/O}=0,$ and therefore (1)

$$\vec{V}_{C/O}=\left(\frac{d\vec{r}_{B/A}}{dt}\right)_{/O_{xyz}}+\left(\frac{d\vec{r}_{C/B}}{dt}\right)_{/O_{xyz}}$$

For emphasis the right hand side has been written as the time derivative of the position vectors $\mathbf{r_{B/O}}$ and $\mathbf{r_{C/B}}$. Because the origins of frames at O and A are coincident, then $\mathbf{r_{B/O}}=\mathbf{r_{A/O}}+\mathbf{r_{B/A}}= \mathbf{0}+ \mathbf{r_{B/A}}$ because $\mathbf{r_{A/O}}=0$.

The time derivatives in the above expression must be taken in the inertial frame O_{xyz} as shown in the above equation. Since both of these position vectors are rotating, then the velocity expressions for both will have components of the form $\vec{\omega}\times\vec{r}$. These will be expanded in complete detail in this early problem set solution so as to emphasize that the derivatives of rotating vectors always have a component due to the rotation of the vector and a component associated with the change in the magnitude of the vector. Taking them one at a time:

$$\vec{V}_{B/A}=\left(\frac{d\vec{r}_{B/A}}{dt}\right)_{/O_{xyz}}=\left(\frac{\delta\vec{r}_{B/A}}{\delta t}\right)_{/A_{x1y1z1}}+\vec{\omega}_{1/O}\times\vec{r}_{B/A}=\left(\vec{V}_{B/A}\right)_{A_{x1y1z1}}+\vec{\omega}_{1/O}\times\vec{r}_{B/A}=0+\omega_1\hat{k}_1\times R_1\hat{i}_1=\omega_1R_1\hat{j}_1$$

$$\vec{V}_{C/B}=\left(\frac{d\vec{r}_{C/B}}{dt}\right)_{/O_{xyz}}=\left(\frac{\delta\vec{r}_{C/B}}{\delta t}\right)_{/B_{x2y2z2}}+\vec{\omega}_{2/O}\times\vec{r}_{C/B}=\left(\vec{V}_{C/B}\right)_{B_{x2y2z2}}+\vec{\omega}_{2/O}\times\vec{r}_{C/B}=0+\left(\omega_1\hat{k}_1+\omega_2\hat{k}_2\right)\times R_2\hat{i}_2$$

$$\vec{V}_{C/B}=\left(\omega_1+\omega_2\right)\hat{k}\times R_2\hat{j}_2=-\left(\omega_1+\omega_2\right)R_2\hat{i}_2$$

where it must be used that $\hat{\mathrm{k}}=\hat{\mathrm{k}}_1=\hat{k}_2$

(2)

The notation system here uses the “/” slash symbol to mean with respect to. Therefore the expression $\left(\frac{\delta\vec{r}_{C/B}}{\delta t}\right)_{/B_{x2y2z2}}=\left(\vec{V}_{C/B}\right)_{/B_{x2y2z2}}=0$ means the partial time derivative of $\mathbf{r_{C/B}}$ with respect to the rotating frame B_{x2y2z2}. It is shown as a partial derivative to emphasize that it accounts for the part of the total time derivative in the inertial frame that comes from the velocity of the point C as seen from within the rotating frame B_{x2y2z2}. Since the frame is attached to the rotating body any velocity component due to the rotation of the body will not be observable. What is observable is any movement of point C relative to the body. Since point C is fixed in this case to the body this relative velocity is zero. Expressed as a relative velocity, $\left(\vec{V}_{C/B}\right)_{/B_{x2y2z2}}=0$.

Similarly the velocity component of point B with respect to A may be written as $\left(\vec{V}_{B/A}\right)_{/A_{x1y1z1}}=\left(\frac{\delta\vec{r}_{B/A}}{\delta t}\right)_{/A_{x1y1z1}}=0$ It is also equal to zero, because B is fixed to the platform and does not move with respect to the rotating a_{a1y1z1} frame.

Therefore the velocities we are attempting to compute come only from the $\vec{\omega}\times\vec{r}$ terms.

$$\vec{V}_{B/O} = \vec{V}_{B/A} = \omega_1 R_1 \hat{j}_1$$

$$\vec{V}_{C/B} = \left(\omega_1 + \omega_2\right) R_2 \hat{j}_2 \tag{3}$$

Adding the two components together yields an expression for the $\vec{V}_{C/O}$:

$$\vec{V}_{C/O} = \vec{V}_{B/A} + \vec{V}_{C/B} = \omega_1 R_1 \hat{j}_1 + \left(\omega_1 + \omega_2\right) R_2 \hat{j}_2$$

An important subtlety is the use of the correct rotation rate. Note that in equation (2) the rotation rate vectors in each case are specified as being the rotation rate with respect to O_{xyz}, the inertial frame. The platform rotates at $\vec{\omega}_{1/O} = \omega_1 \hat{k}_1$. This is with respect to the inertial frame. The arm BC rotates at $\omega_{2/A_{x1y1z1}} = \omega_2 \hat{k}_2$. This rotation rate is specified as the relative rotation rate between the arm and the platform. In evaluating this $\vec{\omega}\times\vec{r}$ term one must use the rotation rate relative to the inertial frame, which in this case is the sum of the two rotation rates; $\vec{\omega}_{2/O} = \omega_1 \hat{k}_1 + \omega_2 \hat{k}_2$.

In this problem the z, z_1 and z_2 axes are all parallel to one another. Therefore they have the same unit vector. That is to say, $\hat{k} = \hat{k}_1 = \hat{k}_2$.

The final expression for the velocity of the passenger at C involves unit vectors from two different reference frames:

$$\vec{V}_{C/O} = \omega_1 R_1 \hat{j}_1 + \left(\omega_1 + \omega_2\right) R_2 \hat{j}_2$$

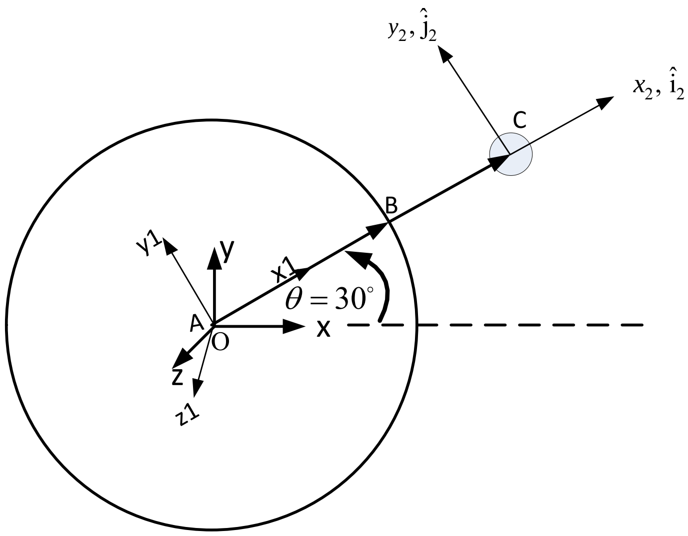
In order to go any further with the problem the actual angular position of the two reference frames must be evaluated and the unit vectors transformed to one common set of unit vectors, for example the unit vectors of the inertial frame.

For example: Let the x_1 axis be at an angle of +30 degrees with respect to the x axis. Let the arm BC be aligned with the radius AB as shown in the following figure. It immediately follows that the unit vectors are related in the following way:

$$\hat{j}_1 = \hat{j}_2 = \hat{i}\cos(30^\circ) + \hat{j}\sin(30) = \frac{\sqrt{3}}{2}\hat{i} + \frac{\hat{j}}{2}$$

From this we can express $\mathbf{V_{C/O}}$ in terms of the units vectors in the inertial frame.

$$\vec{V}_{C/O} = \omega_1 R_1 \hat{j}_1 + \left(\omega_1 + \omega_2\right) R_2 \hat{j}_2 = [(R_1 + R_2)\omega_1 + R_2\omega_2](\frac{\sqrt{3}}{2}\hat{i} + \frac{\hat{j}}{2})$$



Evaluation of the acceleration of point C, $\mathbf{a}_{C/O}$. With the above discussion to inform us, the evaluation of the acceleration proceeds in much the same way. There are just more terms to evaluate.

From the reading on expressing velocities and accelerations in translating and rotating frames, comes the following expression for acceleration in full 3-dimensional vector form.

$$\vec{a}_{B/O} = \vec{a}_{A/O} + (\vec{a}_{B/A})_{/A_{x1y1z1}} + \dot{\omega}\times\vec{r}_{B/A} + \vec{\omega}_{/O}\times(\vec{\omega}_{/O}\times\vec{r}_{B/A}) + 2\vec{\omega}_{/O}\times(\vec{V}_{B/A})_{/A_{x1y1z1}}$$

$$\vec{a}_{B/O} = (1) + (2) + (3) + (4) + (5) \qquad (4)$$

$(1)\equiv$ Translational acceleration of frame A_{x1y1z1} with respect to the inertial frame O_{xyz}.

$(2)\equiv$ Relative acceleration of the point B as seen from within the moving frame.

$(3)\equiv$ Eulerian acceleration due to the angular acceleration of the rotating body.

$(4)\equiv$ Centripetal acceleration term due to the body's rotational velocity.

$(5)\equiv$ Coriolis acceleration due to translatlational velocity of an bject at B relative to the rotating rigid body.

This solution will be done in such way as to show you how to accumulate the terms when there is a cascade of rotating frames, each one connected to the next.

We begin by computing $\mathbf{a}_{B/O}$. B is s fixed point on the platform and is a fixed point in the frame A_{x1y1z1} which rotates with the platform. This is a straight forward application of the formula in (4) above. Begin by identifying all the terms known to be zero.

$\vec{a}_{A/O} = 0$, because the A_{x1y1z1} frame rotates about point O.

$(\vec{a}_{B/A})_{/A_{x1y1z1}} = 0$, because B is a fixed point in the A_{x1y1z1} frame.

$\dot{\omega}_1 = 0$, no rotational acceleration of the platform.

$(V_{B/A})_{/A_{x1y1z1}} = 0$, because B is a fixed point in the A_{x1y1z1} frame.

Therefore $\vec{a}_{B/O} = 0 + 0 + 0 + \omega_1\hat{k}\times(\omega_1\hat{k}\times R_1\hat{i}_1) + 0 = -\omega_1^2 R_1\hat{i}_1$

Next we set out to find the acceleration of point C with respect to point B so that we can complete the computation of $\mathbf{a_{C/O}}$.

$$\vec{a}_{C/O} = \vec{a}_{B/O} + \vec{a}_{C/B}$$

$$\vec{a}_{C/B} = (\vec{a}_{C/B})_{/Bx2y2z2} + \dot{\vec{\omega}}_{2/O} \times \vec{r}_{C/B} + \vec{\omega}_{2/O} \times (\vec{\omega}_{2/O} \times \vec{r}_{C/B}) + 2\vec{\omega}_{2/O} \times (\vec{V}_{C/B})_{/B_{x2y2z2}}$$

where $(\vec{a}_{C/B})_{/Bx2y2z2} = 0$, $\dot{\vec{\omega}}_{2/O} = 0$, and $(\vec{V}_{C/B})_{/B_{x2y2z2}} = 0$

$\vec{\omega}_{2/O} = \vec{\omega}_{2/B_{x2y2z2}} + \vec{\omega}_{1/O_{xyz}} = \omega_2\hat{k} + \omega_1\hat{k} = (\omega_2 + \omega_1)\hat{k}$ and therefore

$$\vec{a}_{C/B} = 0 + 0 + (\omega_2 + \omega_1)\hat{k} \times ((\omega_2 + \omega_1)\hat{k} \times R_2\hat{i}_2) + 0 = -(\omega_2 + \omega_1)^2 R_2\hat{i}_2$$

Putting it all together

$$\vec{a}_{C/O} = \vec{a}_{B/O} + \vec{a}_{C/B} = -\omega_1^2 R_1\hat{i}_1 - (\omega_2 + \omega_1)^2 R_2\hat{i}_2$$

This method can be generalized to compute the velocity or acceleration of a point which is attached to a cascade of rigid bodies: each one rotating with respect to the one it is connected to. By starting at the inertial frame and working ones way out, the velocity of each velocity or acceleration that one computes can become the reference point for the kinetics of the next point further out the chain.

As was found for the velocity computation, this acceleration is expressed in the unit vectors of the rotating frames. To reduce them to the inertial frame the transformations from moving unit vectors to the inertial ones must be invoked. This is done here for the example in which the two moving frames align with one another and both make an angle with respect to the inertial frame of 30 degrees. Then we can express $\mathbf{a_{C/O}}$ as:

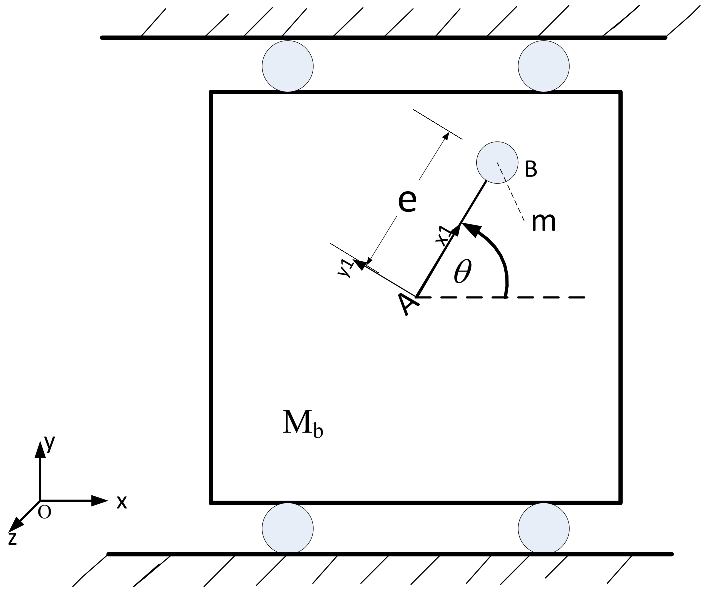
$$\vec{a}_{C/O} = -\omega_1^2 R_1\hat{i}_1 - (\omega_2 + \omega_1)^2 R_2\hat{i}_2 = -(\omega_1^2 R_1 + (\omega_2 + \omega_1)^2 R_2)\hat{i}_2$$

$$\vec{a}_{C/O} = -[\omega_1^2 R_1 + (\omega_2 + \omega_1)^2 R_2][\cos(\theta)\hat{i} + \sin(\theta)\hat{j}] = -[\omega_1^2 R_1 + (\omega_2 + \omega_1)^2 R_2][\frac{\sqrt{3}}{2}\hat{i} + \frac{\hat{j}}{2}]$$

Problem 5: Cart and roller system: A block of mass M_b is constrained to horizontal motion by rollers. Let the acceleration of the block in the inertial frame O_{xyz} be designated as $\vec{a}_{A/O} = \ddot{x}\hat{i}$ which is assumed to be known. Point A is fixed to the block. A massless rod rotates about point A with an angular rate of $\dot{\theta}$. A rotating reference frame, A_{x1y1z1} is attached to the rod, with origin A at the axis of rotation of the rod. The unit vectors in the x_1 and y_1 are $\mathbf{i_1}$ and $\mathbf{j_1}$. A point mass, m, is attached to the end of the rotating rod at point B. The distance from A to B is „e", which is known in the rotating machinery world as the eccentricity. In vector terms $\mathbf{r_{A/B}}$=$\mathbf{e i_1}$.

a. Find an expression for the total acceleration of the mass, m, in the fixed inertial frame O_{xyz}. That is find $\mathbf{a_{B/O}}$. Express the result in terms of x and y components in the O_{xyz} frame. In other words use the unit vectors associated with the O_{xyz} frame. Remember to include the contribution from $\vec{a}_{A/O} = \ddot{x}\hat{i}$ in your answer.

b. What is the magnitude and direction of the axial force that the rod applies to the mass, m? What is the force and direction that the rod places on the pivot point at A?



Problem 5

a. Solution:
We begin by recalling the formula for the acceleration of an object using translating and rotating reference frames.

$$\vec{a}_{B/O} = \vec{a}_{A/O} + (\vec{a}_{B/A})_{/A_{x1y1z1}} + \dot{\omega} \times \vec{r}_{B/A} + \vec{\omega}_{/O} \times (\vec{\omega}_{/O} \times \vec{r}_{B/A}) + 2\vec{\omega}_{/O} \times (\vec{V}_{B/A})_{/A_{x1y1z1}}$$

Those parts known to be zero are: $(\vec{a}_{B/A})_{/A_{x1y1z1}}$, $\dot{\omega}$, and $(\vec{V}_{B/A})_{/A_{x1y1z1}}$

which leads to

$$\vec{a}_{B/O} = \ddot{x}\hat{i} + \vec{\omega}_{/O} \times (\vec{\omega}_{/O} \times \vec{r}_{B/A}) = \ddot{x}\hat{i} + \dot{\theta}\hat{k}_1 \times (\dot{\theta}\hat{k}_1 \times e\hat{i}_1) = \ddot{x}\hat{i} - e\dot{\theta}^2\hat{i}_1$$

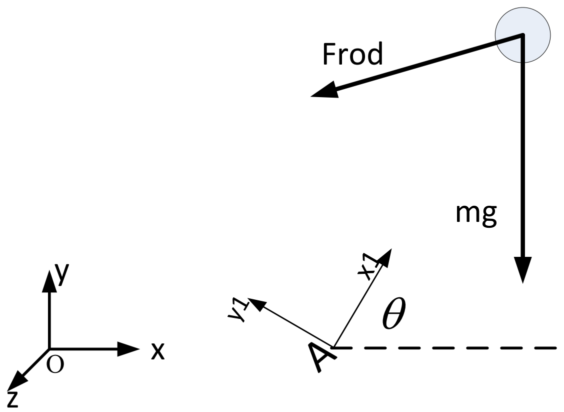
applying the transformation of coordinates: $\hat{\mathrm{i}}_1 = \hat{i}\cos(\theta) + \hat{j}\sin(\theta)$

$$\vec{a}_{B/O} = \ddot{x}\hat{i} - e\dot{\theta}^2[\hat{i}\cos(\theta) + \hat{j}\sin(\theta)]$$

The horizontal and vertical components of the acceleration are given by:

$$\vec{a}_{B/O} = [\ddot{x} - e\dot{\theta}^2\cos(\theta)]\hat{i} - e\dot{\theta}^2\sin(\theta)\hat{j}$$

b. To obtain the force that the massless rod puts on the rotating mass m requires that Newton‟s 2nd law be satisfied. This requires a free body diagram for the mass „m‟.



c. Newton‟s 2nd law requires that

$$\sum \vec{F}_m = m\vec{a}_{B/O} = m[(\ddot{x} - e\dot{\theta}^2 \cos(\theta))\hat{i} - e\dot{\theta}^2 \sin(\theta)\hat{j}] = \vec{F}_{rod} - mg\hat{j}$$

Therefore: $\vec{F}_{rod} = mg\hat{j} + m[(\ddot{x} - e\dot{\theta}^2 \cos(\theta))\hat{i} - e\dot{\theta}^2 \sin(\theta)\hat{j}]$

$$\vec{F}_{rod} = +m[\ddot{x} - e\dot{\theta}^2 \cos(\theta)]\hat{i} + m[g - e\dot{\theta}^2 \sin(\theta)]\hat{j}$$

Newton‟s 3rd law tells us that if the rod puts a force on the small mass then it must put an equal and opposite force on the pivot at A.

$$\vec{F}_A = -\vec{F}_m$$

Assume for a moment that the mass M_b is not allowed to move in the x direction. It is a machine fixed in place. Assume also that we can ignore gravity(this would be the case if the axis of rotation were vertical, for example). Then the force the rotor puts on the main mass M_b is simply given by:

$$\vec{F}_{M_b} = -m\vec{a}_{B/O} = me\dot{\theta}^2[\cos(\theta))\hat{i} + \sin(\theta)\hat{j}] = me\omega^2[\cos(\theta))\hat{i} + \sin(\theta)\hat{j}]$$

This is an important result for all mechanical machines which have rotating parts. If the axis of rotation does not pass through the center of mass of the rotating component, then the rotating component places a force on the bearings given by

$$\vec{F}_{M_b} = me\omega^2[\cos(\theta))\hat{i} + \sin(\theta)\hat{j}]$$

Where „e‟ is the distance from the axis of rotation to the center of mass of the rotating component. Great effort is taken to balance rotating machinery. Everything from car tires to jet engines receive attention to eliminate problems associated with unbalanced rotating parts.

When the unbalance is large, the machine will eventually beat itself to death, as in this video of the washing machine.