

# 18.S190: Introduction to Metric Spaces

Lecturer: Paige Bright

IAP 2023

## 4 January 19, 2023

### Compact Metric Spaces

Last time, we showed that a set in  $\mathbb{R}^n$  is sequentially compact if and only if it is topologically compact, by showing

sequentially compact  $\iff$  closed and bounded  $\stackrel{\text{Heine-Borel}}{\iff}$  topologically compact.

However, by the previous remark, we don't have Heine-Borel for arbitrary metric spaces. Which begs the question: is sequentially compact the same as topologically compact in metric spaces? The answer is yes. To prove this, we first show a handful of preliminary results.

#### Lemma 1 (Lebesgue Number Lemma)

Let  $(X, d)$  be a sequentially compact metric space and  $\{U_i\}_{i \in I}$  be an open cover of  $X$ . Then, there exists an  $r > 0$  such that for each  $x \in X$ ,  $B_r(x) \subseteq U_i$  for some  $i \in I$ .

**Proof:** Before proving this, try to visualize the result!

We prove this lemma through contradiction. Assume that for some  $r > 0$  there exists an  $x \in X$  (possibly depending on  $r$ ) such that for each  $i \in I$ ,  $B_r(x) \not\subseteq U_i$ . Consider the sequence  $\{x_n\}_n$  in  $X$  such that  $B_{1/n}(x_n) \not\subseteq U_i$  for all  $i \in I$ .

Given that  $X$  is sequentially compact,  $\{x_n\}$  has a convergent subsequence  $\{x_{n_k}\}_k$ . Let  $x_{n_k} \rightarrow x \in X$ . Given that  $\{U_i\}$  is an open cover of  $X$ , there exists a  $U_{i_0}$  such that  $x \in U_{i_0}$ . Given  $U_{i_0}$  is open, it also follows that there exists an  $r_0$  such that  $B_{r_0}(x) \subseteq U_{i_0}$ . Hence, choose  $N$  large enough such that  $d(x, x_N) < \frac{r_0}{2}$  and  $\frac{1}{N} < \frac{r_0}{2}$ . Then, if  $y \in B_{1/N}(x_N)$ , then

$$d(x, y) \leq d(x, x_N) + d(x_N, y) < r_0.$$

Therefore,  $y \in B_{r_0}(x) \subseteq U_{i_0}$ . Hence,

$$B_{1/N}(x_N) \subseteq B_{r_0}(x) \subseteq U_{i_0}$$

which is a contradiction. □

We call this  $r$  the **Lebesgue number** of the open cover of  $X$ , which is useful in other applications.

#### Definition 2

A metric space  $X$  is **totally bounded** if, for every  $\epsilon > 0$ , there exists  $x_1, x_2, \dots, x_k \in X$  with  $k$  finite such that  $\{B_\epsilon(x_i) \mid 1 \leq i \leq k\}$  is an open cover of  $X$ .

### Lemma 3

A metric space  $X$  being sequentially compact implies that  $X$  is totally bounded.

**Proof:** Assume that  $X$  is sequentially compact and not totally bounded. Therefore, there exists an  $\epsilon > 0$  such that  $X$  cannot be covered by a collection of open sets of only finitely many  $\epsilon$ -balls. Hence, let  $x_1 \in X$ ,  $x_2 \in X \setminus B_\epsilon(x_1)$ , then  $x_3 \in X \setminus B_\epsilon(x_1) \setminus B_\epsilon(x_2)$  and so on. We know that there exists such  $x_i$  by the previous statement. Hence, for all  $i \neq j$ ,  $d(x_i, x_j) \geq \epsilon$ . Therefore,  $\{x_n\}_n$  has no convergent subsequence as if there was a convergent subsequence it would be Cauchy, and the previous line shows that no subsequence of  $\{x_n\}$  will be Cauchy. This is a contradiction to  $X$  being sequentially compact.  $\square$

### Theorem 4

A metric space  $X$  is (topologically) compact if and only if  $X$  is sequentially compact.

**Proof:** We first show that sequentially compact implies topologically compact, applying the two lemmas developed thus far. Let  $X$  be sequentially compact and let  $\{U_i\}_{i \in I}$  be an open cover of  $X$ . By the Lebesgue number lemma, there exists an  $r > 0$  such that for each  $x \in X$ ,  $B_r(x) \subset U_i$  for some  $i \in I$ . Furthermore, by Lemma 5,  $X$  is totally bounded. Hence, there exists  $y_1, \dots, y_k \in X$  such that

$$X \subset B_r(y_1) \cup \dots \cup B_r(y_k).$$

However, for each  $i \in I$ , we have  $B_r(y_i) \subset U_{j(i)}$  for some  $j(i) \in I$ . (This notation just means for each  $i$ , there exists a  $j \in I$  which depends on  $i$  such that  $B_r(y_i) \subseteq U_j$ .) Thus,  $\{U_{j(1)}, \dots, U_{j(k)}\}$  is a finite subcover for  $X$ . Therefore, every open cover of  $X$  has a finite subcover, and thus sequentially compact implies topologically compact.

We now prove the other direction. Assume for the sake of contradiction there there exists a sequence  $\{x_n\}_n$  in  $X$  with no convergent subsequence. Notice that no term in the sequence can appear infinitely many times, as otherwise there would be a trivial subsequence of  $\{x_n\}$ . Hence, we assume without loss of generality that  $x_i \neq x_j$  if  $i \neq j$ . Furthermore, notice then that for every  $n$  there exists an  $\epsilon_n > 0$  such that  $B_{\epsilon_n}(x_n)$  contains no other terms in the sequence. If this wasn't the case, then there would again be a convergent subsequence of  $\{x_n\}_n$ . Therefore, for each  $i$ , there exists an open ball  $U_i$  centered at  $x_i$  such that  $x_j \notin U_i$  for all  $i \neq j$ .

Additionally, consider  $U_0 = X \setminus \{x_n \mid n \in \mathbb{N}\}$ .  $U_0$  is open, as  $U_0^c = \{x_n \mid n \in \mathbb{N}\}$  is closed (it contains all of its limit points). Hence,

$$U_0 \cup \{U_n \mid n \in \mathbb{N}\}$$

is an open cover of  $X$ . However, this open cover has no finite subcover as any finite collection of the cover will fail to include infinitely many terms from the sequence  $\{x_n\}_n$ . This is a contradiction, and thus topologically compact implies sequentially compact.  $\square$

**Remark 5.** Notice that we technically could've used this proof in the previous lecture, but the Heine-Borel Theorem is so vastly important that I decided to do that proof before today's lecture.

We will now start to look at some illuminating applications of compact sets to reach an even more powerful theorem.

### Recall 6

Let  $X, Y$  be metric spaces and  $f : X \rightarrow Y$  be a continuous function. Then, for all  $U$  open in  $Y$ ,  $f^{-1}(U)$  is open in  $X$ .

### Theorem 7

Let  $X, Y$  be metric spaces and  $f : X \rightarrow Y$  be continuous. Given  $K \subseteq X$ ,  $f(K) \subset Y$  is compact.

**Proof:** Let  $\{U_i\}_{i \in I}$  be an open cover of  $f(K)$ . Then, define  $V_i = \{f^{-1}(U_i)\}_{i \in I}$ , which is open as  $f$  is continuous. Therefore,  $\{f^{-1}(U_i)\}_{i \in I}$  is an open cover of  $K$ . Hence, there exists a finite subcover  $\{V_{i_1}, \dots, V_{i_k}\}$  of  $K$  as  $K$  is compact. Thus,  $\{U_{i_1}, \dots, U_{i_k}\} = \{f(V_{i_1}), \dots, f(V_{i_k})\}$  is a finite subcover of  $f(K)$ . Therefore,  $f(K)$  is compact.  $\square$

### Corollary 8

Let  $X$  be a metric space and  $K \subseteq X$ . Then, given a continuous function  $f : X \rightarrow \mathbb{R}$ ,  $f$  obtains a maximum and minimum finite value on  $K$ .

**Proof:** The proof follows from the previous theorem, a problem on PSET 2.  $\square$

### Corollary 9

Sometimes in particular we want to study bounded continuous functions, and the previous corollary gives us a nice property. Given a compact metric space  $X$ , every continuous function on  $f$  is bounded.

**Proof:** Follows immediately.  $\square$

### Theorem 10 (Cantor's Intersection Theorem)

If  $K_1 \supset K_2 \supset K_3 \supset \dots$  is a decreasing sequence of nonempty sequentially compact subsets of  $\mathbb{R}^n$ , then  $\bigcap_{i \geq 1} K_i$  is non-empty.

**Proof:** Choose a sequence  $\{a_n\}_n$  such that  $a_n \in K_n$  for each  $n$ . We know that there exists such an  $a_n$  as each  $K_n$  is nonempty. Then,  $\{a_n\}_n$  is a sequence in  $K_1$ , and thus there exists a convergent subsequence  $\{a_{n_k}\}_k$  such that  $a_{n_k} \rightarrow a \in K_1$ . Furthermore,  $\{a_n\}_{n=2}^\infty$  is a sequence in  $K_2$ , and thus contains a convergent subsequence. Therefore,  $a \in K_2$ . Continuing this process, we get that  $a \in K_i$  for all  $i$ . Thus,  $a \in \bigcap_{i \geq 1} K_i$ .  $\square$

### Definition 11 (Finite Intersection Property)

A collection of closed sets  $\{C_i\}_i$  has the **finite intersection property** if every finite subcollection has a nonempty intersection.

Given Lemma 3 and the Cantor Intersection Theorem, it is clear that there are some relations between compact sets, nonempty intersections of sets, and totally bounded sets. We hence show the following theorem.

### Theorem 12

Given a metric space  $(X, d)$ , the following are equivalent.

- (1)  $X$  is compact.
- (2)  $X$  is sequentially compact.
- (3)  $X$  is Cauchy complete and totally bounded.
- (4) Every collection of closed subsets of  $X$  with the finite intersection property has a non-empty intersection.

We have shown (1)  $\iff$  (2), and thus we show (1)  $\iff$  (4) and (2)  $\iff$  (3) to finish the proof.

**Proof:** (1)  $\implies$  (4): Assume for the sake of contradiction that there exists a collection of closed subsets  $\{C_i\}_{i \in I}$  with the finite intersection property such that  $\bigcap_{i \in I} C_i = \emptyset$ . Given  $C_i$  is closed in  $X$  for all  $i$ ,  $U_i = C_i^c$  is open in  $X$  for each  $i$ . Then,

$$\bigcup_{i \in I} U_i = \bigcup_{i \in I} C_i^c = \left( \bigcap_{i \in I} C_i \right)^c = \emptyset^c = X.$$

Hence, the  $U_i$  cover  $X$ . Given  $X$  is compact, there exists a finite subcover  $\{U_{i_1}, \dots, U_{i_k}\}$  of  $X$ . Thus,

$$X = \bigcup_{n=1}^k U_{i_n} = \left( \bigcap_{n=1}^k U_{i_n}^c \right)^c = \left( \bigcap_{n=1}^k C_{i_n} \right)^c.$$

Therefore,  $\bigcap_{n=1}^k C_{i_n} = \emptyset$  which is a contradiction with the finite intersection property.

(4)  $\implies$  (1): Suppose that  $\{U_i\}_{i \in I}$  is an open cover of  $X$ , and let  $C_i = U_i^c$  for each  $i \in I$ . Assume for the sake of contradiction that no finite subset of the  $U_i$  covers  $X$ . We show that  $C_i$  has the finite intersection property. Assume for the sake of contradiction that  $\{C_{n_1}, \dots, C_{n_k}\}$  satisfies  $C_{n_1} \cap \dots \cap C_{n_k} = \emptyset$ . Then,

$$\bigcup_{i=1}^k U_{n_i} = \left( \bigcap_{i=1}^k U_{n_i}^c \right)^c = \left( \bigcap_{i=1}^k C_{i_k} \right)^c = \emptyset^c = X.$$

This is a contradiction with the assumption that no subset of the  $U_i$  covers  $X$ . Hence,  $\{C_i\}_{i \in I}$  satisfies the finite intersection property. Therefore,  $\{C_i\}_{i \in I}$  has non-empty intersection; i.e.  $\bigcap_{i \in I} C_i \neq \emptyset$ . Then,  $\bigcup_{i \in I} U_i \neq X$ , which is a contradiction to the  $U_i$  being an open cover for  $X$ . Thus, every open cover of  $X$  has a finite open subcover.

(2)  $\implies$  (3): We have already shown that  $X$  being sequentially compact implies totally bounded, and hence we only need show that sequentially compact implies Cauchy complete. Let  $\{x_n\}$  be a Cauchy sequence in  $X$ . Given  $\{x_n\}$  is a sequence in  $X$ , there exists a convergent subsequence  $\{x_{n_k}\}$  in  $X$  such that  $x_{n_k} \rightarrow x \in X$ . Let  $\epsilon > 0$ , and choose  $N$  such that  $d(x_i, x_j) < \epsilon/2$  whenever  $i, j \geq N$ . Next, choose  $n_k > N$  such that  $d(x_{n_k}, x) < \epsilon/2$ . Then,

$$d(x, x_N) \leq d(x, x_{n_k}) + d(x_{n_k}, x_N) < \epsilon.$$

Thus,  $x_n \rightarrow x \in X$  as  $n \rightarrow \infty$ . Therefore, every Cauchy sequence in  $X$  converges to a point in  $X$ . Hence,  $X$  is Cauchy complete.

(3)  $\implies$  (2): This part of the proof is quite difficult. Consider a sequence  $\{x_n\}_n$  in  $X$ . Given  $X$  is totally bounded, for every  $n \in \mathbb{N}$ , there exists a finite set of points  $\{y_1^{(n)}, \dots, y_{r(n)}^{(n)}\}$  such that  $X \subset B_{\frac{1}{n}}(y_1^{(n)}) \cup \dots \cup B_{\frac{1}{n}}(y_{r(n)}^{(n)})$ . Define

$$S_n = \{y_1^{(n)}, \dots, y_{r(n)}^{(n)}\}.$$

We want to find a convergent subsequence of  $\{x_n\}_n$ . We do so by construction. Given  $S_1$  is finite, there exists a  $y_{n(1)}^{(1)} \in S_1$  such that  $B_1(y_{n(1)}^{(1)})$  contains infinitely many points from  $\{x_n\}_n$ . Choose  $z_1$  from this ball. Then, given  $S_2$  is finite, there is a  $y_{n(2)}^{(2)}$  such that infinitely many points from  $\{x_n\}_n$  are in  $B_1(y_{n(1)}^{(1)}) \cap B_{1/2}(y_{n(2)}^{(2)})$ . Choose  $z_2$  from this set. Continue this procedure for each  $k > 1$ , selecting a  $z_k$  from  $\bigcap_{i=1}^k B_{\frac{1}{i}}(y_{n(i)}^{(i)})$ . Then, we show  $\{z_n\}_n$  is Cauchy. Let  $\epsilon > 0$ . Then, there exists an  $N \in \mathbb{N}$  such that  $\frac{1}{N} < \epsilon$ . Hence, for all  $n, m \geq N$ ,

$$d(z_n, z_m) < \frac{1}{N} < \epsilon.$$

Therefore, by the Cauchy completeness of  $X$ ,  $\{z_n\}$  converges to a point in  $X$ . □

**Remark 13.** Where do we use the fact that each ball has infinitely many points? We do in fact use this property in

*the proof. Try to figure out how!*

MIT OpenCourseWare  
<https://ocw.mit.edu>

18.S190 Introduction to Metric Spaces  
Independent Activities Period (IAP) 2023

For information about citing these materials or our Terms of Use, visit: <https://ocw.mit.edu/terms>.