

18.906: Problem Set I

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet.

Each problem is designed to illuminate material from a lecture, and the problems are numbered to match the lectures.

Extra credit for finding mistakes and telling me about them early!

1. (a) Show that any limit can be expressed as an equalizer of two maps between products.

(b) Let \mathcal{C} and \mathcal{D} be two categories and $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ two functors. In class we said that an adjunction between F and G is an isomorphism

$$\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$$

that is natural in both variables. Show that this is equivalent to giving natural transformations

$$\alpha_X: X \rightarrow GFX, \quad \beta_Y: FGY \rightarrow Y,$$

such that

$$\beta_{FX} \circ F\alpha_X = 1_{FX}, \quad G\beta_Y \circ \alpha_{GY} = 1_{GY}.$$

(c) Suppose that F and F' are both left adjoint to $G: \mathcal{D} \rightarrow \mathcal{C}$. Show that there is a unique natural isomorphism $F \rightarrow F'$ that is compatible with the adjunction.

2. (a) We have used the notation Z^X for a mapping object in a Cartesian closed category, and $\mathcal{C}^{\mathcal{I}}$ for the category of functors to \mathcal{C} from a small category \mathcal{I} . Does this constitute a conflict of notation? Explain.

(b) Let \mathcal{C} be a Cartesian closed category.

(i) Verify the exponential laws: construct natural isomorphisms

$$Z^{X \times Y} \cong (Z^X)^Y, \quad (Y \times Z)^X \cong Y^X \times Z^X.$$

The first of these shows that the adjunction bijection $\mathcal{C}(X \times Y, Z) \cong \mathcal{C}(Y, Z^X)$ “enriches” to an isomorphism in \mathcal{C} . The second says that the product in \mathcal{C} is actually an “enriched” product.

(ii) Construct a “composition” natural transformation

$$Y^X \times Z^Y \rightarrow Z^X$$

using the evaluation maps, and show that it is associative and unital.

(c) Construct left and right adjoints to the forgetful functor

$$u: \mathbf{Top} \rightarrow \mathbf{Set},$$

and conclude that for any small category I , the limit and the colimit of a functor $X : I \rightarrow \mathbf{Top}$ consists of the corresponding limit or colimit of underlying sets endowed with a suitable topology.

(d) Show that the colimit (in \mathbf{Top}) of any diagram of k -spaces is again a k -space, and serves as the colimit in $k\mathbf{Top}$. (Suggestion: Show that in \mathbf{Top} any coproduct of k -spaces is a k -space and that any quotient of a k -space is a k -space, and then use the dual of 1(a).)

3. (a) Show that the smash product is associative as a functor $k\mathbf{Top}_* \times k\mathbf{Top}_* \rightarrow k\mathbf{Top}_*$.

(b) Let W be a pointed k -space. Show that the functors

$$W \wedge - : k\mathbf{Top}_* \rightarrow k\mathbf{Top}_*$$

and

$$(-)_*^W : k\mathbf{Top}_* \rightarrow k\mathbf{Top}_*$$

are *homotopy functors*: they descend to well-defined functors

$$W \wedge - : \mathrm{Ho}(k\mathbf{Top}_*) \rightarrow \mathrm{Ho}(k\mathbf{Top}_*)$$

and

$$(-)_*^W : \mathrm{Ho}(k\mathbf{Top}_*) \rightarrow \mathrm{Ho}(k\mathbf{Top}_*).$$

Hint: Construct a map $A \wedge (X_*^W) \rightarrow (A \wedge X)_*^W$. (Careful about the definition of X_*^W : if (god forbid) the basepoint of X is not closed, this may not be a closed subspace of X^W and so may not be a k -space. But we know how to fix that: k -ify it!)

4. Show that the fiber bundle $SO(n) \rightarrow S^{n-1}$ sending an orthogonal matrix with determinant 1 to its first column has a section if and only if S^{n-1} is parallelizable. What is the situation for $n = 3$? for $n = 4$?

5. Let $p : E \rightarrow B$ and $p' : E' \rightarrow B$ be fibrations, and let $f : E \rightarrow E'$ be a homotopy equivalence such that $p' \circ f = p$. Show that f is in fact a fiber-homotopy equivalence.

In particular, suppose that B is contractible. Then the identity map 1_B is homotopic to a map factoring through $*$. The pullback of E along this trivial map is $B \times p^{-1}(*)$, while the pullback along the identity is E . So any fibration over a contractible space is fiber homotopical equivalent to a product projection.

Hint: First show that it suffices to find a map $g : E' \rightarrow E$ such that $p \circ g = p'$ and $f \circ g$ is fiber-homotopic to $1_{E'}$.

Then reduce this to the following (where E will be what used to be E' , and f is something else again): Suppose that $p : E \rightarrow B$ is a fibration and that $f : E \rightarrow E$ is such that $pf = p$ and $f \sim 1_E$. Then there is a map $g : E \rightarrow E$ such that $pg = p$ and fg is fiber homotopic to the identity.

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18.906 Algebraic Topology II
Spring 2020

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