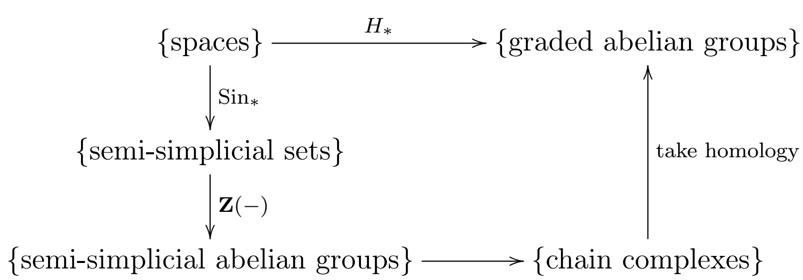
2 Homology

In the last lecture we introduced the standard n-simplex $\Delta^n \subseteq \mathbf{R}^{n+1}$. Singular simplices in a space X are maps $\sigma\colon \Delta^n \to X$ and constitute the set $\mathrm{Sin}_n(X)$. For example, $\mathrm{Sin}_0(X)$ consists of points of X. We also described the face inclusions $d^i : \Delta^{n-1} \to \Delta^n$, and the induced "face maps"

$$d_i : \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n-1}(X)\,, 0 \leq i \leq n\,,$$

given by precomposing with face inclusions: $d_i\sigma = \sigma \circ d^i$. For homework you established some quadratic relations satisfied by these maps. A collection of sets $K_n, n \geq 0$, together with maps $d_i : K_n \to K_{n-1}$ related to each other in this way, is a *semi-simplicial set*. So we have assigned to any space X a semi-simplicial set $S_*(X)$.

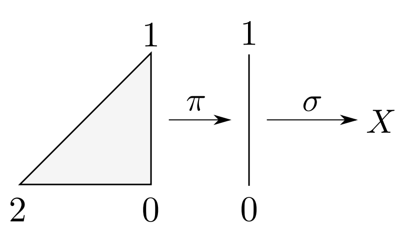
To the semi-simplicial set $\{\mathrm{Sin}_n(X), d_i\}$ we then applied the free abelian group functor, obtaining a semi-simplicial abelian group. Using the d_is, we constructed a boundary map d which makes $S_*(X)$ a *chain complex* – that is, $d^2 = 0$. We capture this process in a diagram:



Example 2.1. Suppose we have $\sigma\colon \Delta^1 \to X$. Define $\phi\colon \Delta^1 \to \Delta^1$ by sending $(t, 1-t)$ to $(1-t, t)$. Precomposing σ with ϕ gives another singular simplex $\overline{\sigma}$ which reverses the orientation of σ. It is *not* true that $\overline{\sigma} = -\sigma$ in $S_1(X)$.

However, we claim that $\overline{\sigma} \equiv -\sigma \bmod B_1(X)$. This means that there is a 2-chain in X whose boundary is $\overline{\sigma} + \sigma$. If $d_0\sigma = d_1\sigma$, so that $\sigma \in Z_1(X)$, then $\overline{\sigma}$ and $-\sigma$ are homologous: $[\overline{\sigma}] = -[\sigma]$ in $H_1(X)$.

To construct an appropriate boundary, consider the projection map $\pi : \Delta^2 \to \Delta^1$ that is the affine extension of the map sending e_0 and e_2 to e_0 and e_1 to e_1.



We'll compute $d(\sigma \circ \pi)$. Some of the terms will be constant singular simplices. Let's write $c_x^n : \Delta^n \to X$ for the constant map with value $x \in X$. Then

$$d(\sigma \circ \pi) = \sigma\pi d^0 - \sigma\pi d^1 + \sigma\pi d^2 = \overline{\sigma} - c^1_{\sigma(0)} + \sigma\,.$$

The constant simplex $c^1_{\sigma(0)}$ is an "error term," and we wish to eliminate it. To achieve this we can use the constant 2-simplex $c^2_{\sigma(0)}$ at $\sigma(0)$; its boundary is

$$c^1_{\sigma(0)} - c^1_{\sigma(0)} + c^1_{\sigma(0)} = c^1_{\sigma(0)}\,.$$

So

$$\overline{\sigma} + \sigma = d(\sigma \circ \pi + c^2_{\sigma(0)})\,,$$

and $\overline{\sigma} \equiv -\sigma \bmod B_1(X)$ as claimed.

Some more language: two cycles that differ by a boundary dc are said to be *homologous*, and the chain c is a *homology* between them.

Let's compute the homology of the very simplest spaces, $\varnothing$ and $*$. For the first, $\mathrm{Sin}_n(\varnothing) = \varnothing$, so $S_*(\varnothing) = 0$. Hence $\cdots \to S_2 \to S_1 \to S_0$ is the zero chain complex. This means that $Z_*(\varnothing) = B_*(\varnothing) = 0$. The homology in all dimensions is therefore 0.

For $*$, we have $\mathrm{Sin}_n(*) = \{c^n_*\}$ for all $n \geq 0$. Consequently $S_n(*) = \mathbf{Z}$ for $n \geq 0$ and 0 for $n < 0$. For each i, $d_i c^n_* = c^{n-1}_*$, so the boundary maps $d\colon S_n(*) \to S_{n-1}(*)$ in the chain complex depend on the parity of n as follows:

$$d(c^n_*) = \sum_{i=0}^{n} (-1)^i c^{n-1}_* = \begin{cases} c^{n-1}_* & \text{for } n \text{ even, and} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

This means that our chain complex is:

$$0 \leftarrow \mathbf{Z} \xleftarrow{0} \mathbf{Z} \xleftarrow{1} \mathbf{Z} \xleftarrow{0} \mathbf{Z} \xleftarrow{1} \cdots\,.$$

The boundaries coincide with the cycles except in dimension zero, where $B_0(*) = 0$ while $Z_0(*) = \mathbf{Z}$. Therefore $H_0(*) = \mathbf{Z}$ and $H_i(*) = 0$ for $i \neq 0$.

We've defined homology groups for each space, but haven't yet considered what happens to maps between spaces. A continuous map $f\colon X \to Y$ induces a map $f_*\colon \mathrm{Sin}_n(X) \to \mathrm{Sin}_n(Y)$ by composition:

$$f_* : \sigma \mapsto f \circ \sigma\,.$$

For f_* to be a map of semi-simplicial sets, it needs to commute with face maps: We need $f_* \circ d_i = d_i \circ f_*$. A diagram is said to be *commutative* if all composites with the same source and target are equal, so this equation is equivalent to commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Sin}_n(X) & \xrightarrow{f_*} & \mathrm{Sin}_n(Y) \\ \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i} \\ \mathrm{Sin}_{n-1}(X) & \xrightarrow{f_*} & \mathrm{Sin}_{n-1}(Y)\,. \end{array}$$

Well, $d_i f_* \sigma = (f_*\sigma) \circ d^i = f \circ \sigma \circ d^i$, and $f_*(d_i\sigma) = f_*(\sigma \circ d^i) = f \circ \sigma \circ d^i$ as well. The diagram remains commutative when we pass to the free abelian groups of chains.

If C_* and D_* are chain complexes, a *chain map* $f\colon C_* \to D_*$ is a collection of maps $f_n\colon C_n \to D_n$ such that the following diagram commutes for every n:

$$\begin{array}{ccc} C_n & \xrightarrow{f_n} & D_n \\ \downarrow{\scriptstyle d_C} & & \downarrow{\scriptstyle d_D} \\ C_{n-1} & \xrightarrow{f_{n-1}} & D_{n-1} \end{array}$$

For example, if $f\colon X \to Y$ is a continuous map, then $f_*\colon S_*(X) \to S_*(Y)$ is a chain map as discussed above.

A chain map induces a map in homology $f_* : H_n(C) \to H_n(D)$. The method of proof is a so-called "diagram chase" and it will be the first of many. We check that we get a map $Z_n(C) \to Z_n(D)$. Let $c \in Z_n(C)$, so that $d_C c = 0$. Then $d_D f_n(c) = f_{n-1} d_C c = f_{n-1}(0) = 0$, because f is a chain map. This means that $f_n(c)$ is also an n-cycle, i.e., f gives a map $Z_n(C) \to Z_n(D)$.

Similarly, we get a map $B_n(C) \to B_n(D)$. Let $c \in B_n(C)$, so that there exists $c' \in C_{n+1}$ such that $d_C c' = c$. Then $f_n(c) = f_n d_C c' = d_D f_{n+1}(c')$. Thus $f_n(c)$ is the boundary of $f_{n+1}(c')$, and f gives a map $B_n(C) \to B_n(D)$.

The two maps $Z_n(C) \to Z_n(D)$ and $B_n(C) \to B_n(D)$ quotient to give a map on homology $f_* : H_n(X) \to H_n(Y)$.

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