Chapter 1

Singular homology

1 Introduction: singular simplices and chains

This is a course on algebraic topology. We'll discuss the following topics.

1. Singular homology
2. CW-complexes
3. Basics of category theory
4. Homological algebra
5. The Künneth theorem
6. Cohomology
7. Universal coefficient theorems
8. Cup and cap products
9. Poincaré duality.

The objects of study are of course topological spaces, and the machinery we develop in this course is designed to be applicable to a general space. But we are really mainly interested in geometrically important spaces. Here are some examples.

- The most basic example is *n-dimensional Euclidean space*, $\mathbf{R}^n$.
- The *n-sphere* $S^n = \{x \in \mathbf{R}^{n+1} : |x| = 1\}$, topologized as a subspace of $\mathbf{R}^{n+1}$.
- Identifying antipodal points in S^n gives *real projective space* $\mathbf{RP}^n = S^n/(x \sim -x)$, i.e. the space of lines through the origin in $\mathbf{R}^{n+1}$.
- Call an ordered collection of k orthonormal vectors an *orthonormal k-frame*. The space of orthonormal k-frames in $\mathbf{R}^n$ forms the *Stiefel manifold* $V_k(\mathbf{R}^n)$, topologized as a subspace of $(S^{n-1})^k$.
- The *Grassmannian* $\mathrm{Gr}_k(\mathbf{R}^n)$ is the space of k-dimensional linear subspaces of $\mathbf{R}^n$. Forming the span gives us a surjection $V_k(\mathbf{R}^n) \to \mathrm{Gr}_k(\mathbf{R}^n)$, and the Grassmannian is given the quotient topology. For example, $\mathrm{Gr}_1(\mathbf{R}^n) = \mathbf{RP}^{n-1}$.

All these examples are *manifolds*; that is, they are Hausdorff spaces locally homeomorphic to Euclidean space. Aside from $\mathbf{R}^n$ itself, the preceding examples are also compact. Such spaces exhibit a hidden symmetry, which is the culmination of 18.905: Poincaré duality.

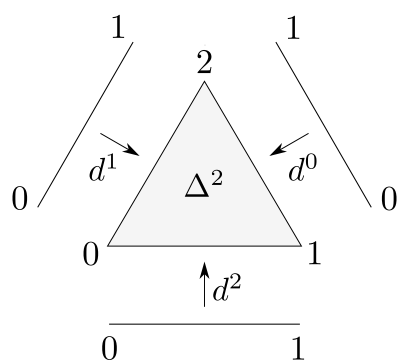
As the name suggests, the central aim of algebraic topology is the usage of algebraic tools to study topological spaces. A common technique is to probe topological spaces via maps to them from simpler spaces. In different ways, this approach gives rise to singular homology and homotopy groups. We now detail the former; the latter takes the stage in 18.906.

Definition 1.1. For $n \geq 0$, the *standard n-simplex* Δ^n is the convex hull of the standard basis $\{e_0, \ldots, e_n\}$ in $\mathbf{R}^{n+1}$:

$$\Delta^n = \left\{ \sum t_i e_i : \sum t_i = 1, t_i \geq 0 \right\} \subseteq \mathbf{R}^{n+1}.$$

The t_i are called *barycentric coordinates.*

The standard simplices are related by face inclusions $d^i \colon \Delta^{n-1} \to \Delta^n$ for $0 \leq i \leq n$, where d^i is the affine map that sends verticies to vertices, in order, and omits the vertex e_i.



Definition 1.2. Let X be any topological space. A *singular n-simplex* in X is a continuous map $\sigma : \Delta^n \to X$. Denote by $\mathrm{Sin}_n(X)$ the set of all n-simplices in X.

This seems like a rather bold construction to make, as $\mathrm{Sin}_n(X)$ is huge. But be patient!

For $0 \leq i \leq n$, precomposition by the face inclusion d^i produces a map $d_i \colon \mathrm{Sin}_n(X) \to \mathrm{Sin}_{n-1}(X)$ sending $\sigma \mapsto \sigma \circ d^i$. This is the "$i$th face" of σ. This allows us to make sense of the "boundary" of a simplex, and we are particularly interested in simplices for which that boundary vanishes.

For example, if σ is a 1-simplex that forms a closed loop, then $d_1\sigma = d_0\sigma$. To express the condition that the boundary vanishes, we would like to write $d_0\sigma - d_1\sigma = 0$ – but this difference is no longer a simplex. To accommodate such formal sums, we will enlarge $\mathrm{Sin}_n(X)$ further by forming the free abelian group it generates.

Definition 1.3. The abelian group $S_n(X)$ of *singular n-chains* in X is the free abelian group generated by n-simplices

$$S_n(X) = \mathbf{Z}\mathrm{Sin}_n(X).$$

So an n-chain is a finite linear combination of simplices,

$$\sum_{i=1}^{k} a_i \sigma_i \,, \quad a_i \in \mathbf{Z} \,, \quad \sigma_i \in \mathrm{Sin}_n(X) \,.$$

If $n < 0$, $\mathrm{Sin}_n(X)$ is declared to be empty, so $S_n(X) = 0$.

We can now define the *boundary operator*

$$d\colon \mathrm{Sin}_n(X) \to S_{n-1}(X),$$

by

$$d\sigma = \sum_{i=0}^{n} (-1)^i d_i \sigma.$$

This extends to a homomorphism $d\colon S_n(X) \to S_{n-1}(X)$ by additivity.

We use this homomorphism to obtain something more tractable than the entirety of $S_n(X)$. First we restrict our attention to chains with vanishing boundary.

Definition 1.4. An *n-cycle* in X is an n-chain c with $dc = 0$. Notation:

$$Z_n(X) = \ker(d : S_n(X) \to S_{n-1}(X)) .$$

For example, if σ is a 1-simplex forming a closed loop, then $\sigma \in Z_1(X)$ since $d\sigma = d_0\sigma - d_1\sigma = 0$.

It turns out that there's a cheap way to produce a cycle:

Theorem 1.5. *Any boundary is a cycle; that is,* $d^2 = 0$.

We'll leave the verification of this important result as a homework problem. What we have found, then, is that the singular chains form a "chain complex," as in the following definition.

Definition 1.6. A *graded abelian group* is a sequence of abelian groups, indexed by the integers. A *chain complex* is a graded abelian group $\{A_n\}$ together with homomorphisms $d : A_n \to A_{n-1}$ with the property that $d^2 = 0$.

The group of n-dimensional *boundaries* is

$$B_n(X) = \mathrm{im}(d : S_{n+1}(X) \to S_n(X)) ,$$

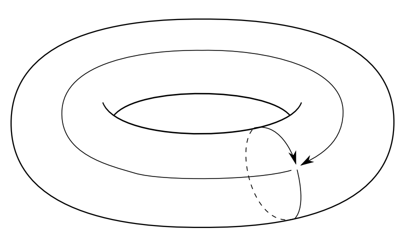
and the theorem tells us that this is a subgroup of the group of cycles: the "cheap" ones. If we quotient by them, what's left is the "interesting cycles," captured in the following definition.

Definition 1.7. The *nth singular homology group* of X is:

$$H_n(X) = \frac{Z_n(X)}{B_n(X)} = \frac{\ker(d : S_n(X) \to S_{n-1}(X))}{\mathrm{im}(d : S_{n+1}(X) \to S_n(X))} .$$

We use the same language for any chain complex: it has cycles, boundaries, and homology groups. The homology forms a graded abelian group.

Both $Z_n(X)$ and $B_n(X)$ are free abelian groups because they are subgroups of the free abelian group $S_n(X)$, but the quotient $H_n(X)$ isn't necessarily free. While $Z_n(X)$ and $B_n(X)$ are uncountably generated, $H_n(X)$ turns out to be finitely generated for the spaces we are interested in. If T is the torus, for example, then we will see that $H_1(T) \cong \mathbf{Z} \oplus \mathbf{Z}$, with generators given by the 1-cycles illustrated below.



We will learn to compute the homology groups of a wide variety of spaces. The n-sphere for example has the following homology groups:

$$H_q(S^n) = \begin{cases} \mathbf{Z} & \text{if} \quad q = n > 0 \\ \mathbf{Z} & \text{if} \quad q = 0, n > 0 \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if} \quad q = n = 0 \\ 0 & \text{otherwise}\,. \end{cases}$$