18.905: Problem Set VI

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet. Scores will be posted on the Stellar website. Extra credit for calling attention to mistakes!

25. $\varnothing$.

26. (Another 3×3 puzzle.) Suppose all the rows and columns in the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

are exact. Construct from this a natural exact sequence

$$0 \to A' \to A \oplus B' \to B \to C'' \to 0\,,$$

It may be easiest to construct two short exact sequences and then splice them together. What is the dual statement?

27. **(a)** Let $0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$ be a short exact sequence of abelian groups, and let C_* be a chain complex of free abelian groups. Construct and prove exactness of the "long exact coefficient sequence"

$$\cdots \longrightarrow H_n(C_* \otimes M') \xrightarrow{i_*} H_n(C_* \otimes M) \xrightarrow{p_*} H_n(C_* \otimes M'') \xrightarrow{\partial} H_{n-1}(C_* \otimes M') \longrightarrow \cdots$$

(b) For example, the short exact sequence $0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$ induces a natural transformation

$$\beta : H_n(X;\mathbb{Z}/2) \longrightarrow H_{n-1}(X;\mathbb{Z}/2)\,.$$

(called the "Bockstein operator"). Show that $\beta^2 = 0 : H_n(X;\mathbb{Z}/2) \longrightarrow H_{n-2}(X;\mathbb{Z}/2)$.

(c) Compute this operator in the mod 2 homology of real projective n space.

28. Recall that $\alpha_p : \Delta^p \to \Delta^n$ is the affine map sending the vertex i to the vertex i (for $0 \le i \le p$), and $\omega_q : \Delta^q \to \Delta^n$ is the affine map sending the vertex j to the vertex $p+j$ (for $0 \le j \le q$). The *Alexander-Whitney map*

$$\alpha_{X,Y} : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$$

sends a simplex $\sigma : \Delta^n \to X \times Y$, with components $\sigma_1 : \Delta^n \to X$ and $\sigma_2 : \Delta^n \to Y$, to the sum

$$\alpha(\sigma) = \sum_{p+q=n} (\sigma_1 \circ \alpha_p) \otimes (\sigma_2 \circ \omega_q)\,.$$

(a) Check that this is a chain map.

(b) Check that it is "associative" and "unital":

$$\begin{array}{ccc} S_*(X \times Y \times Z) & \xrightarrow{\alpha_{X\times Y,Z}} & S_*(X \times Y) \otimes S_*(Z) \\ \downarrow{\scriptstyle \alpha_{X,Y\times Z}} & & \downarrow{\scriptstyle \alpha_{X,Y}\otimes 1} \\ S_*(X) \otimes S_*(Y \times Z) & \xrightarrow{1\otimes\alpha_{Y,Z}} & S_*(X) \otimes S_*(Y) \otimes S_*(Z) \end{array}$$

and

$$\begin{array}{ccc} & S_*(* \times X) & \\ {\scriptstyle c}\swarrow & \downarrow{\scriptstyle \alpha_{*,X}} & \\ S_*(X) & \xleftarrow{\epsilon\cdot 1} & S_*(*) \otimes S_*(X) \end{array} \qquad \begin{array}{ccc} S_*(X \times *) & & \\ \downarrow{\scriptstyle \alpha_{X,*}} & \searrow{\scriptstyle c} & \\ S_*(X) \otimes S_*(*) & \xrightarrow{1\cdot\epsilon} & S_*(X) \end{array}$$

Here the map c is induced from one of the projection isomorphisms $* \times X \to X$ or $X \times * \to X$,

$$\epsilon \cdot 1 : S_*(*) \otimes S_*(X) \xrightarrow{\epsilon\otimes 1} \mathbb{Z} \otimes S_*(X) \xrightarrow{\cong} S_*(X)$$

and $1 \cdot \epsilon$ is similar.

(c) Now observe that α is *not* "commutative": give an example to show that

$$\begin{array}{ccc} S_*(X \times Y) & \xrightarrow{S_*(T)} & S_*(Y \times X) \\ \downarrow{\scriptstyle \alpha_{X,Y}} & & \downarrow{\scriptstyle \alpha_{Y,X}} \\ S_*(X) \otimes S_*(Y) & \xrightarrow{\tau} & S_*(Y) \otimes S_*(X) \end{array}$$

cannot commute for either sign in $\tau(x \otimes y) = \pm y \otimes x$.

(d) But show that this diagram *is* naturally homotopy commutative, with

$$\tau(x \otimes y) = (-1)^{pq} y \otimes x\,, \quad x \in S_p(X)\,, \quad y \in S_q(Y)\,.$$

29. Construct a "semi-relative cross product," natural in X and the pair (Y, B):

$$\times : H^p(X; R) \otimes_R H^q(Y, B; R) \to H^{p+q}(X \times Y, X \times B; R)$$

that agrees with the cross product we constructed in class if $B = \varnothing$ and that makes (R coefficients understood)

$$\begin{tikzcd} H^p(X) \otimes H^q(B) \arrow[r, "\times"] \arrow[d, "1\otimes\partial"] & H^{p+q}(X \times B) \arrow[d, "\partial"] \\ H^p(X) \otimes H^{q+1}(Y,B) \arrow[r, "\times"] & H^{p+q+1}(X \times Y, X \times B) \end{tikzcd}$$

commute, at least up to sign. Give conditions under which it is an isomorphism.

30. Let $A \subseteq X$ and $B \subseteq Y$ be subsets. Construct a natural chain map

$$S_*(X,A) \otimes S_*(Y,B) \to S_*(X \times Y, A \times Y \cup X \times B)$$

that is a homology isomorphism if A and B are open. (Hint: Problem **26.**, or its proof, might be useful.) So there is a natural "relative cross product" map

$$H_*(X,A;R) \otimes_R H_*(Y,B;R) \to H_*(X \times Y, A \times Y \cup X \times B; R)$$

that is an isomorphism if A and B are open, R is a PID, and either $H_*(X,A;R)$ or $H_*(Y,B;R)$ is free over R.

31. **(a)** What is the kth Betti number of $(S^1)^n$?

(b) Define an equivalence relation on $\mathbb{R}^n$ by saying that two vectors are equivalent if they differ by a vector with entries in $\mathbb{Z}$. Identify the quotient space of $\mathbb{R}^n$ by this equivalence relation with the product space $(S^1)^n$. Let M be an $n \times n$ matrix with entries in $\mathbb{Z}$. It defines a linear map $\mathbb{R}^n \to \mathbb{R}^n$ in the usual way. Show that this map descends to a self-map of $(S^1)^n$. Compute the effect of this map on $H_n((S^1)^n)$.