30 Surfaces and nondegenerate symmetric bilinear forms

We are aiming towards a proof of a fundamental cohomological property of manifolds.

Definition 30.1. A (topological) manifold is a Hausdorff space such that every point has an open neighborhood that is homeomorphic to some (finite dimensional) Euclidean space.

If all these Euclidean spaces can be chosen to be $\mathbf{R}^n$, we have an n-manifold.

In this lecture we will state a case of the Poincaré duality theorem and study some consequences of it, especially for compact 2-manifolds. This whole lecture will be happening with coefficients in $\mathbf{F}_2$.

Theorem 30.2. *Let M be a compact manifold of dimension n. There exists a unique class $[M] \in H_n(M)$, called the* fundamental class*, such that for every p, q with $p+q=n$ the pairing*

$$H^p(M) \otimes H^q(M) \xrightarrow{\cup} H^n(M) \xrightarrow{\langle -,[M]\rangle} \mathbf{F}_2$$

is perfect.

This means that the adjoint map

$$H^p(M) \to \mathrm{Hom}(H^q(M), \mathbf{F}_2)$$

is an isomorphism. Since cohomology vanishes in negative dimensions, one thing this implies is that $H^p(M) = 0$ for $p > n$. Since M is compact, $\pi_0(M)$ is finite, and

$$H^n(M) = \mathrm{Hom}(H^0(M), \mathbf{F}_2) = \mathrm{Hom}(\mathrm{Map}(\pi_0(M), \mathbf{F}_2), \mathbf{F}_2) = \mathbf{F}_2[\pi_0(M)]\,.$$

A vector space V admitting a perfect pairing $V \otimes W \to \mathbf{F}_2$ is necessarily finite dimensional; so $H^p(M)$ is in fact finite-dimensional for all p.

Combining this pairing with the universal coefficient theorem, we get isomorphisms

$$H^p(M) \xrightarrow{\cong} \mathrm{Hom}(H^p(M), \mathbf{F}_2) \xleftarrow{\cong} H_q(M)\,.$$

The homology and cohomology classes corresponding to each other under this isomorphism are said to be "Poincaré dual."

Using these isomorphisms, the cup product pairing can be rewritten as a homology pairing:

$$\begin{tikzcd}
H_p(M) \otimes H_q(M) \arrow[r, "\pitchfork"] \arrow[d, "\cong"] & H_{n-p-q}(M) \arrow[d, "\cong"] \\
H^{n-p}(M) \otimes H^{n-q}(M) \arrow[r, "\cup"] & H^{2n-p-q}(M) \,.
\end{tikzcd}$$

This is the *intersection pairing*. Here's how to think of it. Take homology classes $\alpha \in H_p(M)$ and $\beta \in H_q(M)$ and represent them (if possible!) as the image of the fundamental classes of submanifolds of M, of dimensions p and q. Move them if necessary to make them intersect "transversely." Then their intersection will be a submanifold of dimension $n-p-q$, and it will represent the homology class $\alpha \pitchfork \beta$.

This relationship between the cup product and the intersection pairing is the source of the symbol for the cup product.

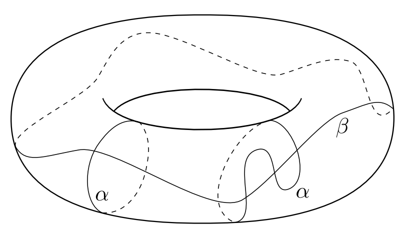
Example 30.3. Let $M = T^2 = S^1 \times S^1$. We know that

$$H^1(M) = \mathbf{F}_2\langle a, b\rangle$$

and $a^2 = b^2 = 0$, while $ab = ba$ generates $H^2(M)$. The Poincaré duals of these classes are represented by cycles α and β wrapping around one or the other of the two factor circles. They can be made to intersect in a single point. This reflects the fact that

$$\langle a \cup b, [M]\rangle = 1 \,.$$

Similarly, the fact that $a^2 = 0$ reflects the fact that its Poincaré dual cycle α can be moved so as not to intersect itself. The picture below shows two possible α's.



This example exhibits a particularly interesting fragment of the statement of Poincaré duality: In an even dimensional manifold – say $n = 2k$ – the cup product pairing gives us a nondegenerate symmetric bilinear form on $H^k(M)$. As indicated above, this can equally well be considered a bilinear form on $H_k(M)$, and it is then to be thought of as describing the number of points (mod 2) two k-cycles intersect in, when put in general position relative to one another. It's called the *intersection form*. We'll denote it by

$$\alpha \cdot \beta = \langle a \cup b, [M]\rangle \,,$$

where again a and α are Poincaré dual, and b and β are dual.

Example 30.4. In terms of the basis α, β, the intersection form for T^2 has matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

This is a "hyperbolic form."

Let's discuss finite dimensional nondegenerate symmetric bilinear forms over $\mathbf{F}_2$ in general. A form on V restricts to a form on any subspace $W \subseteq V$, but the restricted form may be degenerate. Any subspace has an *orthogonal complement*

$$W^\perp = \{v \in V : v \cdot w = 0 \text{ for all } w \in W\}.$$

Lemma 30.5. *The restriction of a nondegenerate bilinear form on V to a subspace W is nondegenerate exactly when $W \cap W^\perp = 0$. In that case $W^\perp$ is also nondegenerate, and the splitting*

$$V \cong W \oplus W^\perp$$

respects the forms.

Using this easy lemma, we may inductively decompose a general (finite dimensional) symmetric bilinear form. First, if there is a vector $v \in V$ such that $v \cdot v = 1$, then it generates a nondegenerate subspace and

$$V = \langle v \rangle \oplus \langle v \rangle^\perp.$$

Continuing to split off one-dimensional subspaces brings us to the situation of a nondegenerate symmetric bilinear form such that $v \cdot v = 0$ for every vector. Unless $V = 0$ we can pick a nonzero vector v. Since the form is nondegenerate, we may find another vector w such that $v \cdot w = 1$. The two together generate a 2-dimensional hyperbolic subspace. Split it off and continue. We conclude:

Proposition 30.6. *Any finite dimensional nondegenerate symmetric bilinear form over $\mathbf{F}_2$ splits as an orthogonal direct sum of forms with matrices* $[1]$ *and* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Let **Bil** be the set of isomorphism classes of finite dimensional nondegenerate symmetric bilinear forms over $\mathbf{F}_2$. We've just given a classification of these things. This is a commutative monoid under orthogonal direct sum. It can be regarded as the set of nonsingular symmetric matrices modulo the equivalence relation of "similarity": Two matrices M and N are *similar* if $N = AMA^T$ for some nonsingular A.

Claim 30.7.

$$\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

Proof. This is the same thing as saying that $\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} = AA^T$ for some nonsingular A. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. □

It's easy to see that there are no further relations; **Bil** is the commutative monoid with two generators I and H, subject to the relation $I + H = 3I$.

Let's go back to topology. Let $n = 2$. Then you get an intersection pairing on $H_1(M)$. Consider $\mathbf{RP}^2$. We know that $H_1(\mathbf{RP}^2) = \mathbf{F}_2$. This must be the form we labelled I. This says that anytime you have a nontrivial cycle on a projective plane, there's nothing you can do to remove its self interesections. You can see this. The projective plane is a Möbius band with a disk sown on along the boundary. The waist of the Möbius band serves as a generating cycle. The observation is that if this cycle is moved to intersect itself tranversely, it must intersect itself an odd number of times.

We can produce new surfaces from old by a process of "addition." Given two connected surfaces Σ_1 and Σ_2, cut a disk out of each one and sew them together along the resulting circles. This is the *connected sum* $\Sigma_1 \# \Sigma_2$.

Proposition 30.8. *There is an isomorphism*

$$H^1(\Sigma_1 \# \Sigma_2) \cong H^1(\Sigma_1) \oplus H^1(\Sigma_2)$$

compatible with the intersection forms.

Proof. Let's compute the cohomology of $\Sigma_1 \# \Sigma_2$ using Mayer-Vietoris. The two dimensional cohomology of $\Sigma_i - D^2$ vanishes because the punctured surface retracts onto its 1-skeleton. The relevant fragment is

$$0 \to H^1(\Sigma_1 \# \Sigma_2) \to H^1(\Sigma_1 - D^2) \oplus H^1(\Sigma_2 - D^2) \to H^1(S^1) \xrightarrow{\delta} H^2(\Sigma_1 \# \Sigma_2) \to 0\,.$$

The boundary map must be an isomorphism, because the connected sum is a compact connected surface so has nontrivial H^2. We leave the verification that the direct sum is orthogonal to you. □

Write **Surf** for the set of homeomorphism classes of compact connected surfaces. Connected sum provides it with the structure of a commutative monoid. The classification of surfaces may now be summarized as folows:

Theorem 30.9. *Formation of the intersection bilinear form gives an isomorphism of commutative monoids* **Surf** $\to$ **Bil***.*

This is a kind of model result of algebraic topology! – a complete algebraic classification of a class of geometric objects. The oriented surfaces correspond to the bilinear forms of type gH; g is the *genus*. But it's a little strange. We must have a relation corresponding to $H \oplus I = 3I$, namely

$$T^2 \# \mathbf{RP}^2 \cong (\mathbf{RP}^2)^{\#3}\,.$$

You should verify this for yourself!

There's more to be said about this. Away from characteristic 2, symmetric bilinear forms and quadratic forms are interchangeable. But over $\mathbf{F}_2$ you can ask for a quadratic form q such that

$$q(x+y) = q(x) + q(y) + x \cdot y\,.$$

This is a "quadratic refinement" of the symmetric bilinear form. Of course it implies that $x \cdot x = 0$ for all x, so this will correspond to some further structure on an oriented surface. This structure is a "framing," a trivialization of the normal bundle of an embedding into a high dimensional Euclidean space. There are then further invariants of this framing; this is the story of the Kervaire invariant.

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