Plane Crystallographic Groups with Point Group D_1.

This note describes discrete subgroups G of isometries of the plane P whose translation lattice L contains two independent vectors, and whose point group $\overline{G}$ is the dihedral group D_1, which consists of the identity and a reflection about the origin. Among the ten possible point groups C_n or D_n with $n = 1, 2, 3, 4, 6$, the analysis of D_1 is among the most complicated. There are three different types of group with this point group.

Let G be a group of the type that we are considering. We choose coordinates so that the reflection in $\overline{G}$ is about the horizontal axis. As in the text, we put bars over symbols that represent elements of the point group $\overline{G}$ to avoid confusing them with the elements of G. So we denote the reflection in $\overline{G}$ by $\overline{r}$.

The lattice L consists of the vectors v such that t_v is in G, and we know that elements of $\overline{G}$ map L to L. If v is in L, $\overline{r}v$ is also in L.

I. The shape of the lattice

Proposition 1. *There are horizontal and vertical vectors $a = (a_1, 0)^t$ and $b = (0, b_2)^t$ respectively, such that, with $c = \frac{1}{2}(a + b)$, L is one of the two lattices L_1 or L_2, where*

$$\begin{aligned} L_1 &= \mathbb{Z}a + \mathbb{Z}b, && \textit{is a 'rectangular' lattice, and} \\ L_2 &= \mathbb{Z}a + \mathbb{Z}c, && \textit{is a 'triangular' lattice.} \end{aligned}$$

Since $b = 2c - a$, $L_1 \subset L_2$.

The lattice L_1 is called 'rectangular' because the horizontal and vertical lines through its points divide the plane into rectangles. The lattice L_2 is obtained by adding to L_1 the midpoints of every one of these rectangles. There are two scale parameters in the description of L – the lengths of the vectors a and b. The usual classification of discrete groups disregards these parameters, but the rectangular and isoceles lattices are considered different.

Proof of the proposition. Let $v = (v_1, v_2)^t$ be an element L not on either coordinate axis. Then $\overline{r}v = (v_1, -v_2)^t$ is in L. So are the vectors $v + \overline{r}v = (2v_1, 0)^t$, and $v - \overline{r}v = (0, 2v_2)^t$. These are nonzero horizontal and vertical vectors in L, respectively.

We choose a_1 to be the smallest positive real number such that $a = (a_1, 0)^t$ is in L. This is possible because L contains a nonzero horizontal vector and it is a discrete group. Then the horizontal vectors in L will be integer multiples of a. We choose b_2 similarly, so that the vertical vectors in L are the integer multiples of $b = (0, b_2)^t$, and we let L_1 be the rectangular lattice $\mathbb{Z}a + \mathbb{Z}b = \{am + bn \mid m, n \in \mathbb{Z}\}$. Then $L_1 \subset L$.

Suppose that $L \neq L_1$. We choose a vector $v = (v_1, v_2)^t$ in L and not in L_1. It will be a linear combination of the independent vectors a and b, say $v = ax + by = (a_1x, b_2y)^t$, with $x, y \in \mathbb{R}$. We write $x = m + x_0$ with $m \in \mathbb{Z}$ and $0 \leq x_0 < 1$, and we write $y = bn + y_0$ in the analogous way. Then $v = (am + bn) + (ax_0 + by_0)$. The vector $am + bn$ is in L_1. We subtract this vector from v, and are reduced to the case that $v = ax + by$, with $0 \leq x, y < 1$. As we saw above, $(2v_1, 0)^t$ is in L. Since this is a horizontal vector, $2v_1$ is an integer multiple of a_1, and since $0 \leq v_1 < a_1$, there are only two possibilities: $v_1 = 0$ or $\frac{1}{2}a_1$. Similarly, $v_2 = 0$ or $v_2 = \frac{1}{2}b_2$. Thus v is one of the four vectors $0, \frac{1}{2}a, \frac{1}{2}b, c$. It is not 0 because $v \notin L_1$. It is not $\frac{1}{2}a$ because a is a horizontal vector of minimal length in L, and it is not $\frac{1}{2}b$ because b is a vertical vector of minimal length. Thus $v = c$, and $L = L_2$. □

II. The glides in G.

We recall that the homomorphism $\pi : M \to O_2$ sends an isometry $m = t_v\varphi$ to the orthogonal operator φ. The point group $\overline{G}$ is the image of G in O_2. So there is an element g in G such that $\pi(g) = \overline{r}$, and $g = t_u r$ for some vector $u = (u_1, u_2)^t$. It is important to keep this in mind: Though $t_u r$ is in G, the translation t_u by itself needn't be in G.

Lemma 2. *Let H be the subgroup of translations t_v in G. So $L = \{v \,|\, t_v \in H\}$, and $H = \{t_v \,|\, v \in L\}$.*
(i) *G is the union of two cosets $H \cup Hg$, where g can be any element not in H.*
(ii) *g has the form $t_u r$, where $u + \overline{r}u$ is in the subgroup $a\mathbb{Z}$.*
(iii) *Let L be a lattice of the form L_1 or L_2, and let $H = \{t_v \,|\, v \in L\}$. Let u be a vector such that $u + \overline{r}u$ is in $a\mathbb{Z}$. Then the set $H \cup Hg$ is a discrete subgroup of M.*

proof. **(i)** Let π_G denote the restriction of π to G. The kernel of this homomorphism is the group H, and its image $\overline{G}$ contains two elements. Therefore there are two cosets of H in G.

(ii) We compute, using the formula $rt_u = t_{\overline{r}u} r$:

$$g^2 = t_u r t_u r = t_{u+\overline{r}u} r^2 = t_{u+\overline{r}u} \tag{3}$$

This is an element of G, so $u + \overline{r}u$ is a horizontal vector in L, an integer multiple of a.

The verification of **(iii)** is similar to the computation made in **(ii)**, and we omit it. $\square$

We check that the isometry $g = t_u r$ is a reflection or a glide with horizontal glide line ℓ defined by $x_2 = \frac{1}{2}u_2$:

$$t_u r \begin{pmatrix} x_1 \\ \frac{1}{2}u_2 \end{pmatrix} = t_u \begin{pmatrix} x_1 \\ -\frac{1}{2}u_2 \end{pmatrix} = \begin{pmatrix} x_1 + u_1 \\ \frac{1}{2}u_2 \end{pmatrix}$$

So g is a horizontal glide along ℓ, as asserted. The glide vector is $(u_1, 0)^t$.

Since the glide line ℓ is horizontal, we can shift coordinates to make it the horizontal axis. This changes the vector u, which becomes the horizontal vector $(u_1, 0)^t$. Then $\overline{r}u = u$, and therefore $g^2 = t_{2u}$ (see (3)). So t_{2u} is in G, and $2u$ is in L. Since $2u$ is a horizontal vector, it is an integer multiple of a. We adjust u, multiplying g on the left by a power of t_a to make $u = 0$ or $\frac{1}{2}a$.

The two dichotomies

$$L = L_1 \text{ or } L_2, \quad \text{and} \quad u = 0 \text{ or } \tfrac{1}{2}a,$$

leave us with four possibilities.

To complete the discussion we must decide whether or not such groups exist, and whether they are different. They do exist, because $H \cup Hg$ is a group (Lemma 2 **(iii)**). And the two types of lattice are considered different. But when the element g we have found is a glide, G might still contain a reflection. This happens when $L = L_2$ and $u = \frac{1}{2}a$. In that case, $c = \frac{1}{2}(a + b)$ is in L, and so $t_{-c}g = t_{-\frac{1}{2}b} r$ is an element of G. Because $-\frac{1}{2}b$ is a vertical vector, this motion is a reflection (about the horizontal line $x_2 = \frac{1}{4}b$). Shifting coordinates once more eliminates this case. This phenomenon doesn't happen when $L = L_1$, so we are left with three types of group.

Theorem. *Let G be a discrete group of isometries of the plane whose point group is the dihedral group $D_1 = \{\overline{1}, \overline{r}\}$. Let $H = \{t_v \in G\}$ be its subgroup of translations.*
(i) *The lattice $L = \{v \,|\, t_v \in G\}$ has one of the forms L_1 or L_2 given in Proposition 1.*
(ii) *Let $u = \frac{1}{2}a$ and let $g = t_u r$. Coordinates in the plane can be chosen so that,*
a) if $L = L_1$, $G = H \cup Hr$ or $G = H \cup Hg$, and
b) if $L = L_2$, $G = H \cup Hr$. $\square$