Doob's Optional Stopping Theorem

The Doob's optional stopping time theorem is contained in many basic texts on probability and Martingales. (See, for example, Theorem 10.10 of *Probability with Martingales*, by David Williams, 1991.) The essential content of the theorem is that you can't make money (in expectation) by buying and selling an asset whose price is a martingale. Precisely, the theorem states that if you buy the asset at some time and adopt any strategy at all for deciding when to sell it, then the expected price at the time you sell is the price you originally paid. Thus—if market price is a martingale—you cannot make money in expectation by "timing the market."

Let Ω be the probability space. Let T be a map from Ω to the set of positive integers. We think of $T(\omega)$ as giving the time at which the asset will be sold if the price sequence is $S(0), S(1), S(2), \ldots$ (where each $S(i)$ is a random variable, i.e., a real-valued function of ω). We say that T is a **stopping time** if the event that $T(\omega) = n$ depends only on the values $S(i)$ for $i \leq n$. In other words, the decision of whether to sell the stock at time n depends only on the history of the stock up until time n, and not on the future values of the stock (which the investor hasn't seen yet).

Doob's Optional Stopping Theorem: If the sequence $S(0), S(1), S(2), \ldots$ is a bounded martingale, and T is a stopping time, then the expected value of $S(T)$ is $S(0)$.

Most real world asset prices are not martingales, even in theory. So why is this theorem relevant to finance?

1. Many asset prices behave approximately like martingales in the short term.

2. According to the **fundamental theorem of asset pricing**, as presented in Zastawniak and Capiński (see the text for the precise conditions of the theorem), the discounted price $\frac{S(n)}{A(n)}$, where A is a risk-free asset, *is* a martingale with respected to the **risk neutral probability**.

3. Sequences of conditional expectations of a quantity—involving conditioning on increasing amounts of information—are martingales. For example, let C be the amount of oil available for drilling under a particular piece of land. Suppose that ten geological tests are done

that will ultimately determine the value of C. Let C_n be the **conditional expectation** of C *given* the outcome of the first n of these tests. Then the sequence $C_0, C_1, C_2, \ldots, C_{10} = C$ is a martingale.

SOME MARTINGALE PROBLEMS:

1. Suppose Harriet has 7 dollars. Her plan is to make one dollar bets on fair coin tosses until her wealth reaches either 0 or 50, and then to go home. What is the expected amount of money that Harriet will have when she goes home? What is the probability that she will have 50 when she goes home?

2. Consider a contract that at time N will be worth either 100 or 0. Let $S(n)$ be its price at time $0 \leq n \leq N$. If $S(n)$ is a martingale, and $S(0) = 47$, then what is the probability that the contract will be worth 100 at time N?

3. Pedro plans to buy the contract in the previous problem at time 0 and sell it the first time T at which the price goes above 55 or below 15. What is the expected value of $S(T)$?

4. Suppose $S(N)$ is with probability one either 100 or 0 and that $S(0) = 50$. Suppose further there is at least a sixy percent probability that the price will at some point dip to below 40 and then subsequently rise to above 60 before time N. Prove that $S(n)$ cannot be a martingale.

18.440 Probability and Random Variables
Spring 2014