18.404/6.840 Lecture 19

Last time:

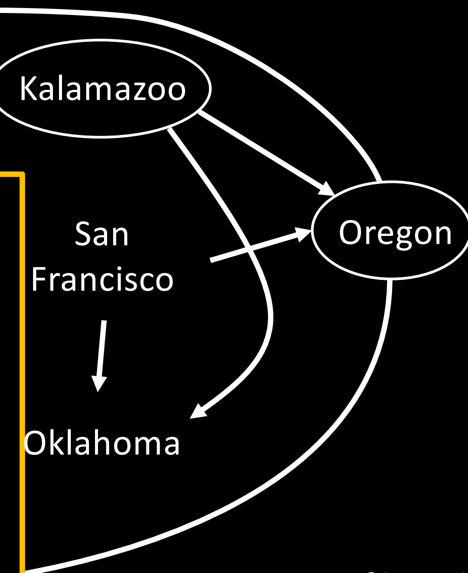
- Review $LADDER_{\mathrm{DFA}} \in$ PSPACE
- Savitch's Theorem: NSPACE$\big(f(n)\big) \subseteq$ SPACE$\big(f^2(n)\big)$
- $TQBF$ is PSPACE-complete

Today: (Sipser §8.3 – §8.4)

- Games and Quantifiers
- The Formula Game
- Generalized Geography is PSPACE-complete
- Logspace: L and NL

Games and Complexity

Geography game



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Check-in 19.1

Let G be the graph below.

Which player has a winning strategy in the Generalized Geography game starting at node a?

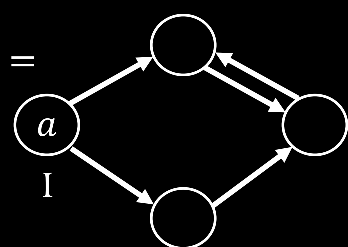
(a) Player I

(b) Player II

(c) Neither player

(d) Both players

$G =$



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which ended the previous place. No repeats allowed.
The first player stuck (= cannot move) loses.

Generalized Geography Game
Played on any directed graph.
Players take turns picking nodes that form a simple path.
The first player stuck loses.

Defn: $GG = \{\langle G, a\rangle \mid$ Player I has a <u>forced win</u> in Generalized Geography on graph G starting at node $a\}$.

"forced win" also called a "winning strategy" means that the player will win if both players play optimally.

Theorem: GG is PSPACE-complete

Check-in 19.1

Games and Quantifiers

The Formula Game

Given QBF $\phi = \exists x_1\ \forall x_2\ \exists x_3\ \cdots (\exists/\forall) x_k\ \overbrace{[\ (\cdots) \wedge \cdots \wedge (\cdots)\ ]}^{\psi}$

There are two Players "∃" and "∀".

Player ∃ assigns values to the ∃-quantified variables.
Player ∀ assigns values to the ∀-quantified variables.
The players choose the values according to the order of the quantifiers in ϕ.

After all variables have been assigned values, we determine the winner:
Player ∃ wins if the assignment satisfies ψ.
Player ∀ wins if not.

Claim: Player ∃ has a forced win in the formula game on ϕ iff ϕ is TRUE.
Therefore $\{\langle\phi\rangle|$ Player ∃ has a forced win on $\phi\} = TQBF$.

Next: show $TQBF \leq_P GG$.

Check-in 19.2
Which player has a winning strategy in the formula game on

$$\phi = \exists x\ \forall y\ [(x \vee y) \wedge (\overline{x} \vee \overline{y})]$$

(a) ∃-player
(b) ∀-player
(c) Neither player

Check-in 19.2

GG is PSPACE-complete

Theorem: GG is PSPACE-complete

Proof: 1) $GG \in$ PSPACE (recursive algorithm, exercise)

2) $TQBF \leq_{\mathrm{P}} GG$

Give reduction f from $TQBF$ to GG. $f(\langle\phi\rangle) = \langle G, a\rangle$

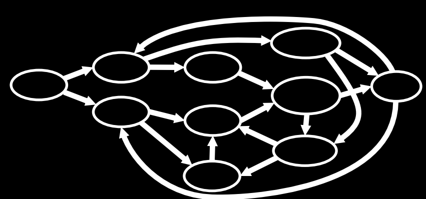
Construct G to mimic the formula game on ϕ.

G has Players I and II

Player I plays role of ∃-Player in ϕ. Ditto for Player II and the ∀-Player.

$$\phi = \exists x_1 \, \forall x_2 \, \exists x_3 \, \cdots (\exists/\forall) x_k \, [\underbrace{(\cdots) \wedge \cdots \wedge (\cdots)}_{\text{assume in cnf}}]$$

$\downarrow f$

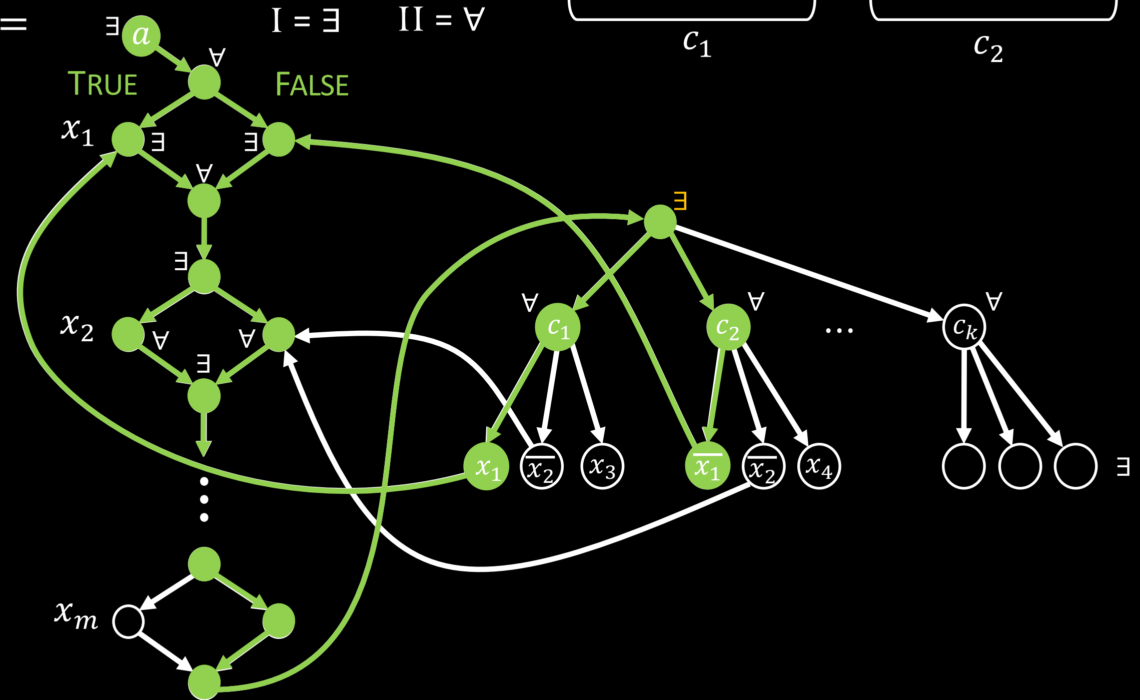
$G =$ 

Constructing the GG graph G

Illustrate construction by example

Say $\phi = \exists x_1 \, \forall x_2 \, \exists x_3 \, \cdots \, \forall x_k \, [\, \underbrace{(\, x_1 \vee \overline{x_2} \vee x_3 \,)}_{c_1} \wedge \underbrace{(\overline{x_1} \vee \overline{x_2} \vee x_4)}_{c_2} \wedge \cdots \wedge \underbrace{(\, \cdots \,)}_{c_k} \,]$

$G =$ $\quad$ $\mathrm{I} = \exists$ $\quad$ $\mathrm{II} = \forall$



Endgame

∃ should win if assignment satisfied all clauses

∀ should win if some unsatisfied clause

Implementation

∀ picks clause node claimed unsatisfied

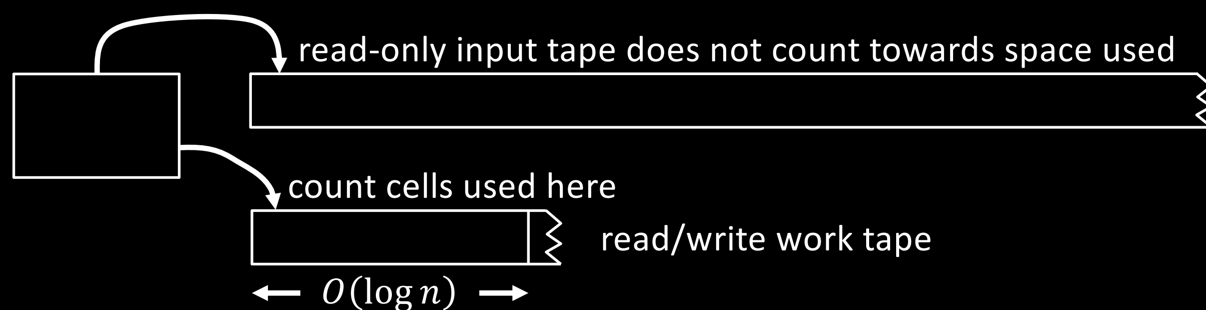
∃ picks literal node claimed to satisfy the clause

liar will be stuck

Log space

To define sublinear space computation, do not count input as part of space used.
Use 2-tape TM model with read-only input tape.

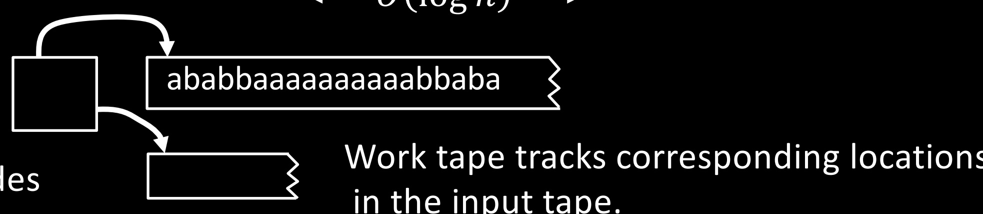
Defn: L = SPACE$(\log n)$
NL = NSPACE$(\log n)$



Log space can represent a constant number of pointers into the input.

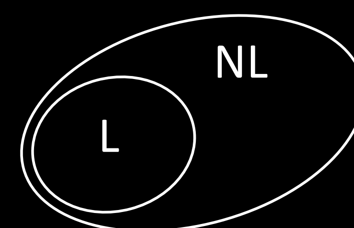
Examples

1. $\{ww^{\mathcal{R}} \mid w \in \Sigma^*\} \in$ L



2. $PATH \in$ NL

Nondeterministically select the nodes of a path connecting s to t.



L = NL? Unsolved

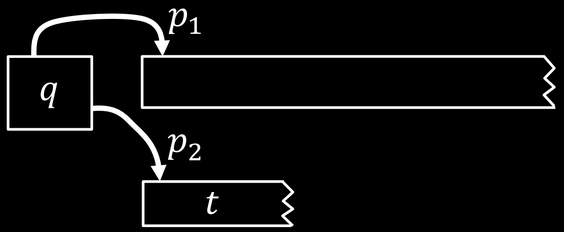
Log space properties

Theorem: L ⊆ P

Proof: Say M decides A in space $O(\log n)$.

Defn: A configuration for M on w is (q, p_1, p_2, t) where q is a state, p_1 and p_2 are the tape head positions, and t is the tape contents.
The number of such configurations is $|Q| \times n \times O(\log n) \times d^{O(\log n)} = O(n^k)$ for some k.

Therefore M runs in polynomial time.
Conclusion: $A \in$ P



Theorem: NL ⊆ SPACE$(\log^2 n)$
Proof: Savitch's theorem works for log space

NL properties

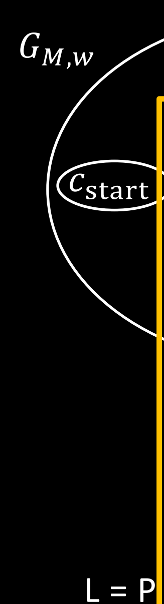
Theorem: NL ⊆ P

Proof: Say NTM M decides A in space $O(\log n)$.

Defn: The <u>configuration graph</u> $G_{M,w}$ for M on w has
  nodes: all configurations for M on w
  edges: edge from $c_i \rightarrow c_j$ if c_i can yield c_j in 1 step.

Claim: M accepts w iff the configuration graph $G_{M,w}$ has a path from c_{start} to c_{accept}

Polynomial time algorithm T for A:
T = "On input w
1. Construct the $G_{M,w}$.
2. *Accept* if there is a path from c_{start} to c_{accept}.
   Reject if not."



Check-in 19.3

We showed that $PATH$ ∈ NL.
What is the best we know about the deterministic space complexity of $PATH$?

(a) $PATH$ ∈ PSPACE
(b) $PATH$ ∈ SPACE(n)
(c) $PATH$ ∈ SPACE$(\log^2 n)$
(d) $PATH$ ∈ SPACE$(\log n)$

Check-in 19.3

Quick review of today

1. The Formula Game
2. Generalized Geography is PSPACE-complete
3. Log space: L and NL
4. Configuration graph
5. NL $\subseteq$ P