12. Instability Dynamics

12.1 Capillary Instability of a Fluid Coating on a Fiber

We proceed by considering the surface tension-induced instability of a fluid coating on a cylindrical fiber.
Define mean thickness

$$h^* = \frac{1}{\lambda}\int_0^\lambda h(x)dx \tag{12.1}$$

Local interfacial thickness

$$h(x) = h^* + \epsilon \cos kx \tag{12.2}$$

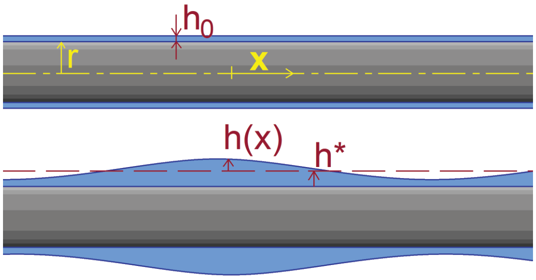

Figure 12.1: Instability of a fluid coating on a cylindrical fiber.

Volume conservation requires:

$$\int_0^\lambda \pi(r+h)^2 dx = \int_0^\lambda \pi(r+h_0)^2 dx \Rightarrow \int_0^\lambda (r+h^*+\epsilon\cos kx)^2 dx = (r+h_0)^2\lambda \Rightarrow$$

$$(r+h^*)^2\lambda + \epsilon^2\frac{\lambda}{2} = (r+h_0)^2\lambda \Rightarrow (r+h^*)^2 = (r+h_0)^2 - \frac{\epsilon^2}{2} = (r+h_0)^2\left[1-\frac{1}{2}\frac{\epsilon^2}{(r+h_0)^2}\right]$$

which implies

$$h^* = h_0 - \frac{1}{4}\frac{\epsilon^2}{r+h_0} \tag{12.3}$$

`Note:`
$h^* < h_0$ which suggests instability.
So, when does perturbation reduce surface energy? i.e. when is $\int_0^\lambda 2\pi(r+h)ds < 2\pi(r+h_0)\lambda$?
`Note:` $ds^2 = dh^2 + dx^2 \Rightarrow ds = dx\sqrt{1+\left(\frac{dh}{dx}\right)} \approx dx\left[1+\frac{1}{2}\epsilon^2k^2\sin^2 kx\right]^{1/2}$

$\int_0^\lambda (r+h)ds = \int_0^\lambda (r+h^*+\epsilon\cos kx)(1+\frac{1}{2}\epsilon^2k^2\sin^2 kx)^{1/2}dx = (r+h^*)\lambda + \frac{1}{4}(r+h^*)\epsilon^2k^2\lambda$.
So the inequality holds provided $(r+h^*)\lambda + \frac{1}{4}(r+h^*)\epsilon^2k^2\lambda < (r+h_0)\lambda$.
Substitute for h^* from (12.3):

$$-\frac{1}{4}\frac{\epsilon^2}{r+h_0} + \frac{1}{4}(r+h^*)\epsilon^2k^2 < 0 \tag{12.4}$$

We note that the result is independent of ϵ:

$$k^2 < (r+h_0)^{-1}(r+h^*)^{-1} \approx \frac{1}{(r+h_0)^2} \tag{12.5}$$

i.e. unstable wavelengths are prescribed by

$$\lambda = \frac{2\pi}{k} > 2\pi(r+h_0) \tag{12.6}$$

as in standard inviscid Ra-P. All long wavelength disturbances will grow. Which grows the fastest? That is determined by the *dynamics* (not just geometry). We proceed by considering the dynamics in the thin film limit, $h_0 \ll r$, for which we obtain the lubrication limit.

12.2 Dynamics of Instability (Rayleigh 1879)

Physical picture: Curvature pressure induced by perturbation drives Couette flow that is resisted by viscosity

$$\eta \frac{d^2 v}{dy} - \frac{dp}{dx} = 0 \tag{12.7}$$

where $\frac{dp}{dx}$ is the gradient in curvature pressure, which is independent of y (a generic feature of lubrication problems), so we can integrate the above equation to obtain

$$v(y) = \frac{1}{\mu}\frac{dp}{dx}\left(\frac{y^2}{2} - hy\right) \tag{12.8}$$

Flux per unit length:

$$Q = \int_0^h v(y)dy = -\frac{1}{3\mu}\frac{dp}{dx}h^3 \tag{12.9}$$

Conservation of volume in lubrication problems requires that $Q(x+dx) - Q(x) = -\frac{\partial h}{\partial t}dx \Rightarrow$

$$\frac{dQ}{dx} = -\frac{h_0^3}{3\mu}\frac{d^2 p}{dx^2} = -\frac{\partial h}{\partial t} \tag{12.10}$$

Curvature pressure

$$p(x) = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \sigma\left(\frac{1}{r+h} - h_{xx}\right) \tag{12.11}$$

Substitute (12.11) into (12.10):

$$\frac{\partial h}{\partial t} = \frac{\sigma h_0^3}{3\mu}\frac{\partial^2}{\partial x^2}\left[\frac{1}{r+h(x)} - \sigma h_{xx}\right] \tag{12.12}$$

Now $h(x,t) = h^* + \epsilon(t)\cos kx \Rightarrow h_x = -\epsilon k \sin kx$, $h_{xx} = -\epsilon^2 k \cos kx$, $h_t = \frac{d\epsilon}{dt}\cos kx$
So $\cos kx \frac{d\epsilon}{dt} = \frac{\sigma h_0^3}{3\mu}\epsilon \cos kx \left[\frac{k^2}{(r+h)^2} - k^4\right] \Rightarrow \frac{d\epsilon}{dt} = \beta\epsilon$ where $\beta = \frac{\sigma h_0^3}{3\mu}\left[\frac{k^2}{(r+h_0)^2} - k^4\right]$
Fastest growing mode when $\frac{d\beta}{dk} = 0 = \frac{2k^8}{(r+h_0)^2} - 4k^{*3}$ so

$$\lambda^* = 2\sqrt{2}\pi\,(r+h_0) \tag{12.13}$$

is the most unstable wavelength for the viscous mode.

Note:

- Recall that for classic Ra-P on a cylindrical fluid thread $\lambda^* \sim 9R$.
- We see here the timescale of instability: $\tau^* = \frac{12\mu(r+h_0)^4}{\sigma h_0^3}$.
- Scaling Argument for Pinch-off time.
  When $h \ll r$, $\nabla p \sim \frac{\sigma h_0}{r^2}\frac{1}{r} \sim \mu \frac{v}{h_0^2} \Rightarrow v \sim \frac{r}{\tau} \sim \frac{h_0^3}{\mu}\frac{\sigma h_0}{r^3} \Rightarrow$

$$\tau_{pinch} \sim \frac{\mu r^4}{\sigma h_0^3} \tag{12.14}$$

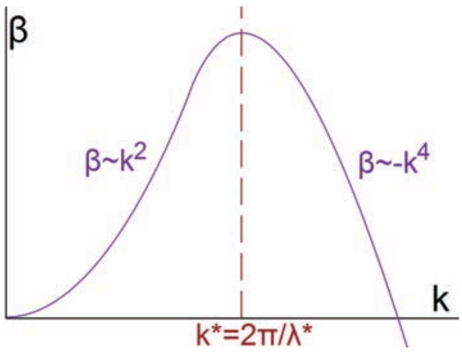

Figure 12.2: Growth rate β as a function of wavenumber k for the system depicted in Fig. 12.1.

12.3 Rupture of a Soap Film (Culick 1960, Taylor 1960)

We assume $\mathcal{O}h = \frac{\mu\nu}{\sigma R} \ll 1$, so that viscous effects are negligible. The driving curvature force is thus resisted principally by fluid inertia. Assume dynamics is largely 2D (true for a planar film, or for bubble burst for $r(t) \gg h$).

Retraction of a Planar Sheet

`Note`: Force/ length acting on the rim may be calculated exactly via Frenet-Serret

$$\boldsymbol{F}_C = \int_C \sigma \left(\boldsymbol{\nabla} \cdot \boldsymbol{n}\right) \boldsymbol{n} dl \tag{12.15}$$

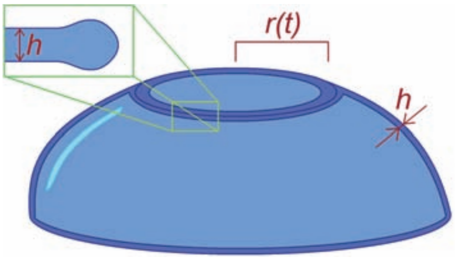

Figure 12.3: Rupture of a soap film of thickness h.

where $\left(\boldsymbol{\nabla} \cdot \boldsymbol{n}\right) \boldsymbol{n} = \frac{d\boldsymbol{t}}{dl}$

$$\Rightarrow \qquad \boldsymbol{F}_C = \int_C \sigma \frac{d\boldsymbol{t}}{dl} dl = \sigma\left(\boldsymbol{t}_1 - \boldsymbol{t}_2\right) = 2\sigma\hat{\boldsymbol{x}} \tag{12.16}$$

At time $t=0$, planar sheet of thickness h punctured at $x=0$, and retracts in $\hat{x}$ direction owing to $\boldsymbol{F}_c$.

`Observation`: The rim engulfs the film, and there is no upstream disturbance.

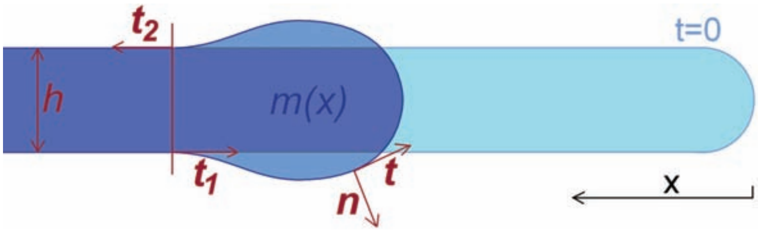

Figure 12.4: Surface-tension-induced retraction of a planar sheet of uniform thickness h released at time $t=0$.

Rim mass: $m(x) = \rho h x$ and speed $v = \frac{dx}{dt}$.

Since the inertial force on the rim is equal to the rate of change of rim momentum

$$F_I = \frac{d}{dt}(mv) = v\frac{d}{dx}mv = v^2\frac{dm}{dx} + mv\frac{dv}{Dx} = \frac{1}{2}v^2\frac{dm}{dx} + \frac{1}{2}\frac{d}{dx}(mv^2)\ . \tag{12.17}$$

The force balance us between the curvature force and the inertial force

$$2\sigma = \frac{d}{dx}(\frac{1}{2}mv^2) + \frac{1}{2}\rho h v^2 \tag{12.18}$$

Integrate from 0 to x:

$$2\sigma x = \frac{1}{2}\rho h x v^2 + \frac{1}{2}\rho h \int_0^x v^2 dx \tag{12.19}$$

The first term is the surface energy released per unit length, the 2nd term the K.E. of the rim, and the 3rd term the energy required to accelerate the rim. Now we assume v is independent of x (as observed in experiments), thus $\int_0^x v^2 dx = xv^2$ and the force balance becomes $2\sigma x = \rho h x v^2 \Rightarrow$

$$v = \left(\frac{2\sigma}{\rho h}\right)^{1/2} \text{ is the retraction speed (Taylor-Culick speed)} \tag{12.20}$$

E.g. for water-soap film, $h \sim 150\mu m \Rightarrow v \sim 10^2$cm/s.

`Note`: Surface area of rim/ length: $p = 2\pi R$ where $m = \rho h x = \pi\rho R^2 \Rightarrow R = \sqrt{\frac{hx}{\pi}}$ where R is the rim radius. Therefore the rim surface energy is $\sigma P = \sigma 2\pi\sqrt{\frac{hx}{\pi}} = 2\sigma\sqrt{hx\pi}$. Total surface energy of the system is $\sigma\left[2x + 2(\pi h x)^{1/2}\right]$.
Scale: $\frac{SA_{rim}}{SA_{sheet}} \sim \frac{2\sqrt{hx\pi}}{2x} \sim \left(\frac{h\pi}{x}\right)^{1/2} \ll 1$ for $x \gg h$.
The rim surface area is thus safely neglected once the sheet has retracted a distance comparable to its thickness.
Some final comments on soap film rupture.

1. for dependence on geometry and influence of μ, see *Savva & Bush (JFM 2009)*.
2. form of sheet depends on $\sqrt{\mathcal{O}h} = \frac{\mu}{\sqrt{2h\rho\sigma}}$.
3. The growing rim at low $\mathcal{O}h$ is subject to Ra-Plateau $\Rightarrow$ scalloping of the retracting rim $\Rightarrow$ rim pinches off into drops
4. At very high speed, air-induced shear stress leads to flapping. The sheet thus behaves like a flapping flag, but with Marangoni elasticity.

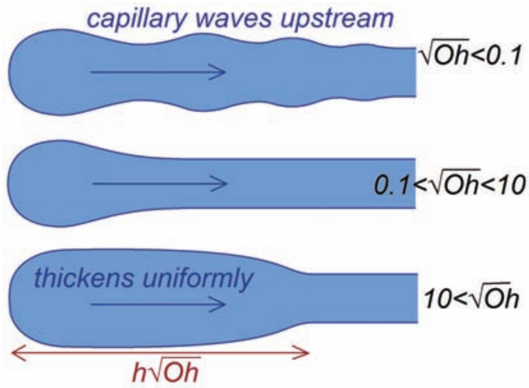

Figure 12.5: The different shapes of a retracting sheet and rim depend on the value of $\mathcal{O}h$.

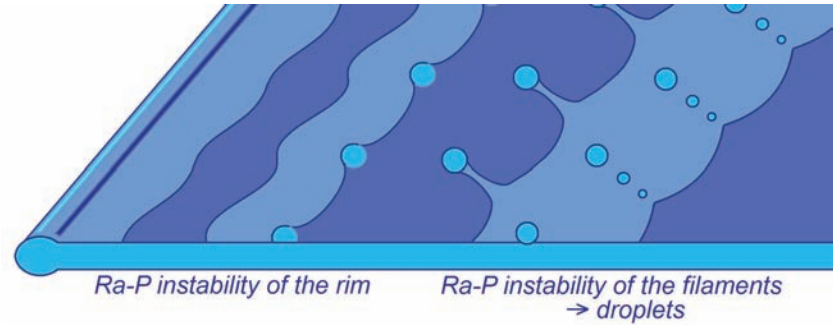

Figure 12.6: The typical evolution of a retracting sheet. As the rim retracts and engulfs fluid, it eventually becomes Rayleigh-Plateau unstable. Thus, it develops variations in radius along its length, and the retreating rim becomes scalloped. Filaments are eventually left by the retracting rim, and pinch off through a Rayleigh-Plateau instability, the result being droplets.

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