22 The Ekman layer

We would now like to return to our coffee cup problem, to get the right answer. To do so, we shall consider the effect of walls on the inviscid flow we calculated in the previous lecture. For starters, lets consider a jar with the top moving at angular velocity Ω_T and the bottom moving at angular velocity Ω_B. Clearly, if $\Omega_T = \Omega_B$ then our inviscid solution applies. Let's try and figure out what happens when Ω_T becomes different from Ω_B.

22.1 A small deviation

Suppose Ω_T=Ω and $\Omega_B = \Omega + \epsilon$. Now there is no way to satisfy the no slip condition on both the top and bottom while having the whole flow spin at angular velocity Ω. Let's move first to the rotating frame, and try to compute the secondary flow that is induced. Clearly, without viscosity it is impossible to solve this problem because the Taylor-Proudman theorem states that inviscid flow is two dimensional (and so no gradient in Ω across the cylinder axis is possible). We therefore anticipate that even though the Rossby number is small, there will be boundary layers. Let's divide the flow into three regions: (1) A boundary layer at the top plate; (2) a boundary layer at the bottom plate; and (3) a central inviscid region.

In the inviscid region we would expect that the solution is (u_I, v_I, w_I), where

$$-2\Omega v_I = -\frac{1}{\rho}\frac{\partial p_I}{\partial x}, \tag{526a}$$

$$2\Omega u_I = -\frac{1}{\rho}\frac{\partial p_I}{\partial y}, \tag{526b}$$

$$0 = \frac{1}{\rho}\frac{\partial p_I}{\partial z}. \tag{526c}$$

In the same way as before, we expect the pressure gradient of the outer flow to force the boundary layer at the rotating wall. Let's consider the structure of the boundary layer at the bottom wall, $z = 0$. There the equations are

$$-2\Omega v = -\frac{1}{\rho}\frac{\partial p_I}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}, \tag{527a}$$

$$2\Omega u = -\frac{1}{\rho}\frac{\partial p_I}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}, \tag{527b}$$

$$0 = \frac{1}{\rho}\frac{\partial p_I}{\partial z} + \nu\frac{\partial^2 w}{\partial z^2}, \tag{527c}$$

$$\nabla\cdot\boldsymbol{u} = 0. \tag{527d}$$

Here we have made the boundary layer approximation that $\partial/\partial z \gg \partial/\partial x, \partial/\partial y$.

From the continuity equation we deduce that w is much smaller than the velocity components parallel to the boundary so that $\partial p_I/\partial z = 0$, and the equations become

$$-2\Omega(v - v_I) = \nu\frac{\partial^2 u}{\partial z^2}, \tag{528a}$$

$$2\Omega(u - u_I) = \nu\frac{\partial^2 v}{\partial z^2}. \tag{528b}$$

These are the equations we must solve. Acheson has a good trick. Multiplying the second equation by i and adding the two yields

$$\nu\frac{\partial^2 f}{\partial z^2} = 2\Omega i f, \tag{529a}$$

where

$$f = u - u_I + i(v - v_I). \tag{529b}$$

The solution is obtained by guessing $f \sim e^{\alpha z}$, which yields $\alpha^2 = 2\Omega i/\nu$. Hence,

$$f = Ae^{(1+i)z^*} + Be^{-(1+i)z^*}, \qquad z^* = z\sqrt{\Omega/\nu}. \tag{530}$$

We require that as $z^* \to \infty$, $f \to 0$. This implies that $A = 0$. We are in the frame of reference moving with the bottom plate, so the no slip boundary condition at $z = 0$ requires that $f(z = 0) = -u_I - iv_I$. Splitting f into its real an imaginary parts implies

$$u = u_I - \mathrm{e}^{-z^*}(u_I \cos(z/\delta) + v_I \sin(z/\delta)), \tag{531}$$
$$v = v_I - \mathrm{e}^{-z^*}(v_I \cos(z/\delta) - u_I \sin(z/\delta)). \tag{532}$$

This is the velocity profile in the boundary layer.

What about the z-component? From the divergence free condition, we have

$$\left(\frac{\Omega}{\nu}\right)^{1/2} \frac{\partial w}{\partial z^*} = \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y}\right) e^{-z^*} \sin z^*. \tag{533}$$

Integrating from $z^* = 0$ to ∞ gives

$$w = \frac{1}{2}\left(\frac{\Omega}{\nu}\right)^{-1/2} \left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y}\right) = \frac{\hat{\omega}_I}{2}\sqrt{\frac{\nu}{\Omega}}, \tag{534}$$

where $\hat{\omega}_I$ is the vorticity in the inviscid flow. Thus, if $\hat{\omega}_I > 0$ (i.e., the bottom boundary is moving slower than the main body of fluid) then there is flow from the boundary layer into the fluid.

22.2 Matching

Now we have these Ekman layers at the top and the bottom. What we just assumed was that the boundary is moving at frequency Ω. If it is not, but instead moving at an angular frequency Ω_B relative to the rotating frame, then we need to change the boundary conditions a little in the rotating frame. In this case

$$w = \left(\frac{\nu}{\Omega_B}\right)^{1/2} \left(\frac{\hat{\omega}_I}{2} - \Omega_B\right). \tag{535a}$$

We could derive this, but it is intuitive since $(\hat{\omega}_I - 2\Omega_B)$ is the vorticity of the interior flow relative to the moving lower boundary. Similarly, if Ω_T denotes the angular velocity of the rigid upper boundary relative to the rotating frame, then there is a small z-component of velocity up into the boundary layer

$$w = \left(\frac{\nu}{\Omega_T}\right)^{1/2} \left(\Omega_T - \frac{\hat{\omega}_I}{2}\right). \tag{535b}$$

Now in our container both are happening. Since u_I, v_I and w_I are all independent of z then so is $\hat{\omega}_I$. Thus, the only way the experiment could work is if the induced value of $\hat{\omega}_I$ from both cases matches. This implies that

$$\hat{\omega}_I = \Omega_T + \Omega_B. \tag{536}$$

With $\Omega_B = 0$ and $\Omega_T = \epsilon$ we have that $\hat{\omega}_I = \epsilon$. Thus, the flow in the inner region has a velocity which is entirely set by the boundary layers. Note that there is no viscosity in this formula, but viscosity plays a role in determining the flow. We have completely different behaviour for $\nu = 0$ and in the limit $\nu \to 0$.

22.3 Spin-down of this apparatus

We now want to finally solve the spin-down of our coffee cup. To do so we assume the coffee cup to be a cylinder with a top and a bottom both rotating with angular velocity $\Omega + \epsilon$. At $t = 0$ the angular velocity of the boundaries is reduced to Ω. How long does it take to reach a steady state?

We use the time-dependent formula

$$\frac{\partial u_I}{\partial t} - 2\Omega v_I = -\frac{1}{\rho}\frac{\partial p_I}{\partial x}, \tag{537a}$$

$$\frac{\partial v_I}{\partial t} + 2\Omega u_I = -\frac{1}{\rho}\frac{\partial p_I}{\partial y}. \tag{537b}$$

Then differentiate the first equation with respect to y and the second with respect to x. Subtracting the latter from the former, and using the continuity equation, we obtain the vorticity equation

$$\frac{\partial \hat{\omega}_I}{\partial t} = 2\Omega \frac{\partial w_I}{\partial z}. \tag{538}$$

Now $\hat{\omega}_I$ is independent of z, so

$$\int_0^L \frac{\partial \hat{\omega}_I}{\partial t} dz = L\frac{\partial \hat{\omega}_I}{\partial t} = 2\Omega[w(L) - w(0)]. \tag{539}$$

The velocity is equal and opposite at the two boundaries (flow is leaving both boundary layers), and has magnitude $(\nu/\Omega)^{1/2}\hat{\omega}_I/2$. Thus

$$\frac{\partial \hat{\omega}_I}{\partial t} = -\frac{2\sqrt{\Omega\nu}}{L}\hat{\omega}_I, \tag{540}$$

implying that vorticity is decreasing in the interior with a characteristic decay time $L/2\sqrt{\Omega\nu}$. For the coffee cup this gives us a much more realistic spin down time compared to our experiments. In real life we should note that diffusion of the no-slip condition also will play a role, and there will be competition between the two depending on the particular shape of your coffee cup. If you go look at the corresponding flow in Acheson, you can now also understand the deep reason why coffee grounds end up at the centre of your cup.

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