21 Rotating flows

In our above discussion of airfoils, we have neglected viscosity which led to d'Alembert's paradox. To illustrate further the importance of boundary layers, we will consider one more example of substantial geophysical importance, where the dynamics of laminar flows is completely controlled by the boundary layer. In the process of deriving this result we will also learn about a rather remarkable phenomenon in rotating fluid dynamics.[27]

21.1 The Taylor-Proudman theorem

Consider a fluid rotating with angular velocity Ω. The equation of motion in the frame of reference rotating with the fluid is

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \Omega \times (\Omega \times \boldsymbol{r}) = -\frac{1}{\rho} \nabla p_{\Omega} + \nu \nabla^2 \boldsymbol{u} - 2\Omega \times \boldsymbol{u}, \tag{515a}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{515b}$$

[27] See Acheson, pp. 278-287

There are two additional terms: the first $\Omega \times (\Omega \times \boldsymbol{r})$ is the centrifugal acceleration, which can be thought of as an augmentation to the pressure distribution, using the identity

$$\Omega \times (\Omega \times \boldsymbol{r}) = -\frac{1}{2}\nabla(\Omega \times \boldsymbol{r})^2. \tag{516}$$

Henceforth, we will simply absorb this into the pressure by writing

$$p = p_\Omega - \frac{\rho}{2}(\Omega \times \boldsymbol{r})^2. \tag{517}$$

For the rotating earth, the effect of this force is to simply distort the shape of the object from a sphere into an oblate ellipsoid. The second term is the *Coriolis acceleration* which is velocity dependent. Hopefully you have heard about it in classical mechanics.

We are going to be interested in flows which are much weaker than the rotation of the system. If U is a characteristic velocity scale and L is a characteristic length scale, then the advective term is of order U^2/L whereas the coriolis force is of order ΩU. We will assume that $\Omega U \gg U^2/L$ so that the equation of motion is effectively

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} - 2\Omega \times \boldsymbol{u}, \tag{518a}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{518b}$$

21.2 Steady, inviscid flow

Now let's consider flow at high Reynolds number. The Reynolds number is now $\Omega L^2/\nu$ within this framework. As usual, the first step is to write down the inviscid equations (since the viscosity is small), and then the next step is to correct them with boundary layers.

Following Acheson, let's write the flow velocity as $\boldsymbol{u} = (u_I, v_I, w_I)$ and $\boldsymbol{\Omega} = (0, 0, \Omega)$. The steady, inviscid flow satisfies

$$2\Omega v_I = \frac{1}{\rho}\frac{\partial p_I}{\partial x}, \tag{519a}$$

$$2\Omega u_I = -\frac{1}{\rho}\frac{\partial p_I}{\partial y}, \tag{519b}$$

$$0 = \frac{1}{\rho}\frac{\partial p_I}{\partial z}, \tag{519c}$$

$$\frac{\partial u_I}{\partial x} + \frac{\partial v_I}{\partial y} + \frac{\partial w_I}{\partial z} = 0. \tag{519d}$$

The third equation says that the pressure is independent of z. Hence, the first two equations say that u_I and v_I are independent of z. Then the last equation says that w_I is independent of z. Thus, *the entire fluid velocity is independent of z* ! This result, which is remarkable, is called the *Taylor-Proudman theorem.* Proudman discovered the theorem, but Taylor discovered what is perhaps its most remarkable consequence.

21.3 Taylor columns

In his paper "Experiments on the motion of solid bodies in rotating fluids", Taylor posed the simple question: given the above fact that slow steady motions of a rotating liquid must be two-dimensional, what happens if one attempts to make a three dimensional motion by, for example, pushing a three dimensional object through the flow with a small uniform velocity? At the beginning of his paper he points out three possibilities:

1. The motion in the liquid is never steady.
2. The motion is steady, but our assumption that u_I is small relative to the rotation velocity breaks down near the object.
3. The motion is steady and two dimensional.

He remarks that the first possibility is unlikely, since it must settle down eventually. The realistic possibilities are (2) and (3). His paper, which can be downloaded from the course page, demonstrates that actually what happens is possibility (3). This is really rather remarkable (as Taylor notes) because there is only one way that it can really happen: An entire column of fluid must move atop the object.

21.4 More on rotating flows

Above, we wrote the equations of a rotating fluid assuming that the rotation frequency dominated the characteristic hydrodynamic flows in the problem. In other words, if Ω is the characteristic rotation frequency, L is a horizontal lengthscale, and U is a typical velocity in the rotating frame, we assumed that

$$\mathrm{Ro} = \frac{U}{\Omega L} \ll 1. \tag{520}$$

This dimensionless number is called the *Rossby number*. The equations were

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} &= -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} - 2\Omega \times \boldsymbol{u}, && (521a) \\ \nabla \cdot \boldsymbol{u} &= 0. && (521b) \end{aligned}$$

We will use these ideas to revisit the famous problem of the spin-down of a coffee cup that we discussed at the very beginning of class. You might recall that the problem we had was that our simple theory of how the spin-down occurred turned out to be entirely false. We shall now construct the correct theory, while learning a bit of atmospheric and oceanic physics along the way.

21.5 More on the Taylor-Proudman theorem

Let's consider the Taylor-Proudman theorem again, this time using another method. We consider the viscosity to be small so that we can use the limit of a stationary, inviscid fluid. More precisely, we consider the ratio

$$\mathrm{E_k} = \frac{\nu \nabla^2 \boldsymbol{u}}{\Omega \times \boldsymbol{u}} = \frac{\nu}{\Omega L^2} \ll 1. \tag{522}$$

This dimensionless number is called the *Ekman number*. The flow is then strictly two-dimensional, and the Navier-Stokes equation simplifies to

$$-\frac{1}{\rho}\nabla p = 2\boldsymbol{\Omega} \times \boldsymbol{u}. \tag{523}$$

Taking the curl of both sides, we find

$$\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) = \boldsymbol{\Omega}\nabla \cdot \boldsymbol{u} - \boldsymbol{u}\nabla \cdot \boldsymbol{\Omega} + \boldsymbol{u} \cdot \nabla\boldsymbol{\Omega} - \boldsymbol{\Omega} \cdot \nabla\boldsymbol{u} = -\Omega\frac{\partial \boldsymbol{u}}{\partial z}. \tag{524}$$

Here, we have used the fact that the fluid velocity is divergence free. Hence we have that $\partial\boldsymbol{u}/\partial z{=}0$, or that the fluid velocity is independent of z. A major consequence of this (Taylor columns) was discussed above.

Before leaving this topic, let's make one more remark. Taking the dot product of the equations of inviscid flow with $\boldsymbol{u}$, we get

$$\boldsymbol{u} \cdot \nabla p = -\rho\boldsymbol{u} \cdot (2\Omega \times \boldsymbol{u}) = 0. \tag{525}$$

This formula states that *the velocity field moves perpendicular to the pressure gradient*, which is somewhat against one's intuition. Hence, the fluid actually moves along lines of constant pressure. Pressure work is not performed either on the fluid or by the fluid. Geophysicists call this fact the *geostrophic balance*.

There is an entertaining fact that one can deduce about atmospheric flows. For an atmospheric flow, the analogue of Ω is not the earth's rotation speed $\hat{\omega}$, but instead $\Omega = \hat{\omega}\sin\phi$, where ϕ is the longitude. Now, this shows that the effective Ω changes sign in the northern and southern hemisphere. What does this imply for the dynamics? When $\Omega > 0$ the velocity moves with the high pressure on the right. Conversely in the southern hemisphere, the velocity moves with the high pressure on the left. It is also true that because of this change in sign, Naval warships have to adjust their range finding when crossing over the equator. However, the myth about the bathtub vortex does not hold because one cannot throw out inertial and viscous terms in solving this problem. The Coriolis force is only important on large scales.

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