

10 Some basic differential geometry

In the next section, we will look more closely at the the mathematical description of elastic materials. To prepare this discussion, it is useful to briefly recall a few basics of differential geometry.

10.1 Differential geometry of curves

Consider a continuous curve $\mathbf{r}(t) \in \mathbb{R}^3$, where $t \in [0, T]$. The length of the curve is given by

$$L = \int_0^T dt \|\dot{\mathbf{r}}(t)\| \quad (222)$$

where $\dot{\mathbf{r}}(t) = d\mathbf{r}/dt$ and $\|\cdot\|$ denotes the Euclidean norm. The local unit tangent vector is defined by

$$\mathbf{t} = \frac{\dot{\mathbf{r}}}{\|\dot{\mathbf{r}}\|}. \quad (223)$$

The unit normal vector, or unit curvature vector, is

$$\mathbf{n} = \frac{(\mathbf{I} - \mathbf{t}\mathbf{t}) \cdot \ddot{\mathbf{r}}}{\|(\mathbf{I} - \mathbf{t}\mathbf{t}) \cdot \ddot{\mathbf{r}}\|}. \quad (224)$$

Unit tangent vector $\hat{\mathbf{t}}(t)$ and unit normal vector $\hat{\mathbf{n}}(t)$ span the *osculating* ('kissing') plane at point t . The unit binormal vector is defined by

$$\mathbf{b} = \frac{(\mathbf{I} - \mathbf{t}\mathbf{t}) \cdot (\mathbf{I} - \mathbf{n}\mathbf{n}) \cdot \ddot{\mathbf{r}}}{\|(\mathbf{I} - \mathbf{t}\mathbf{t}) \cdot (\mathbf{I} - \mathbf{n}\mathbf{n}) \cdot \ddot{\mathbf{r}}\|}. \quad (225)$$

The orthonormal basis $\{\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)\}$ spans the local Frenet frame.

The local curvature $\kappa(t)$ and the associated radius of curvature $\rho(t) = 1/\kappa$ are defined by

$$\kappa(t) = \frac{\dot{\mathbf{t}} \cdot \mathbf{n}}{\|\dot{\mathbf{r}}\|}, \quad (226)$$

and the local torsion $\tau(t)$ by

$$\tau(t) = \frac{\dot{\mathbf{n}} \cdot \mathbf{b}}{\|\dot{\mathbf{r}}\|}. \quad (227)$$

Plane curves satisfy, by definition, $\mathbf{b} = \text{const.}$ or, equivalently, $\tau = 0$.

Given $\|\dot{\mathbf{r}}\|$, $\kappa(t)$, $\tau(t)$ and the initial values $\{\mathbf{t}(0), \mathbf{n}(0), \mathbf{b}(0)\}$, the Frenet frames along the curve can be obtained by solving the Frenet-Serret system

$$\frac{1}{\|\dot{\mathbf{r}}\|} \begin{pmatrix} \dot{\mathbf{t}} \\ \dot{\mathbf{n}} \\ \dot{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \quad (228a)$$

The above formulas simplify if t is the arc length, for in this case $\|\dot{\mathbf{r}}\| = 1$.

As a simple example (which is equivalent to our shortest path problem) consider a polymer confined in a plane. Assume the polymer's end-points are fixed at $(x, y) = (0, 0)$ and $(x, y) = (0, L)$, respectively, and that the ground-state configuration corresponds to a straight line connecting these two points. Denoting the tension⁹ by γ , adopting the parameterization $y = h(x)$ for the polymer and assuming that the bending energy is negligible, the energy relative to the ground-state is given by

$$E = \gamma \left[\int_0^L dx \sqrt{1 + h_x^2} - L \right], \quad (229)$$

where $h_x = h'(x)$. Restricting ourselves to small deformations, $|h_x| \ll 1$, we may approximate

$$E \simeq \frac{\gamma}{2} \int_0^L dx h_x^2. \quad (230)$$

Minimizing this expression with respect to the polymer shape h yields the Euler-Lagrange equation

$$h_{xx} = 0. \quad (231)$$

10.2 Two-dimensional surfaces

We now consider an orientable surface in \mathbb{R}^3 . Possible local parameterizations are

$$\mathbf{F}(s_1, s_2) \in \mathbb{R}^3 \quad (232)$$

where $(s_1, s_2) \in U \subseteq \mathbb{R}^2$. Alternatively, if one chooses Cartesian coordinates $(s_1, s_2) = (x, y)$, then it suffices to specify

$$z = f(x, y) \quad (233a)$$

or, equivalently, the implicit representation

$$\Phi(x, y, z) = z - f(x, y). \quad (233b)$$

The vector representation (232) can be related to the 'height' representation (233a) by

$$\mathbf{F}(x, y) = \begin{pmatrix} x \\ y \\ f(x, y) \end{pmatrix} \quad (234)$$

⁹ γ carries units of energy/length.

Denoting derivatives by $\mathbf{F}_i = \partial_{s_i} \mathbf{F}$, we introduce the surface metric tensor $g = (g_{ij})$ by

$$g_{ij} = \mathbf{F}_i \cdot \mathbf{F}_j, \quad (235a)$$

abbreviate its determinant by

$$|g| := \det g, \quad (235b)$$

and define the associated Laplace-Beltrami operator ∇^2 by

$$\nabla^2 h = \frac{1}{\sqrt{|g|}} \partial_i (g_{ij}^{-1} \sqrt{|g|} \partial_j h), \quad (235c)$$

for some function $h(s_1, s_2)$. For the Cartesian parameterization (234), one finds explicitly

$$\mathbf{F}_x(x, y) = \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix}, \quad \mathbf{F}_y(x, y) = \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix} \quad (236)$$

and, hence, the metric tensor

$$g = (g_{ij}) = \begin{pmatrix} \mathbf{F}_x \cdot \mathbf{F}_x & \mathbf{F}_x \cdot \mathbf{F}_y \\ \mathbf{F}_y \cdot \mathbf{F}_x & \mathbf{F}_y \cdot \mathbf{F}_y \end{pmatrix} = \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_y f_x & 1 + f_y^2 \end{pmatrix} \quad (237a)$$

and its determinant

$$|g| = 1 + f_x^2 + f_y^2, \quad (237b)$$

where $f_x = \partial_x f$ and $f_y = \partial_y f$. For later use, we still note that the inverse of the metric tensor is given by

$$g^{-1} = (g_{ij}^{-1}) = \frac{1}{1 + f_x^2 + f_y^2} \begin{pmatrix} 1 + f_y^2 & -f_x f_y \\ -f_y f_x & 1 + f_x^2 \end{pmatrix}. \quad (237c)$$

Assuming the surface is regular at (s_1, s_2) , which just means that the tangent vectors \mathbf{F}_1 and \mathbf{F}_2 are linearly independent, the local unit normal vector is defined by

$$\mathbf{N} = \frac{\mathbf{F}_1 \wedge \mathbf{F}_2}{\|\mathbf{F}_1 \wedge \mathbf{F}_2\|}. \quad (238)$$

In terms of the Cartesian parameterization, this can also be rewritten as

$$\mathbf{N} = \frac{\nabla \Phi}{\|\nabla \Phi\|} = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}. \quad (239)$$

Here, we have adopted the convention that $\{\mathbf{F}_1, \mathbf{F}_2, \mathbf{N}\}$ form a right-handed system.

To formulate ‘geometric’ energy functionals for membranes, we still require the concept of curvature, which quantifies the local bending of the membrane. We define a 2×2 -curvature tensor $R = (R_{ij})$ by

$$R_{ij} = \mathbf{N} \cdot (\mathbf{F}_{ij}) \quad (240)$$

and local *mean curvature* H and local *Gauss curvature* K by

$$H = \frac{1}{2} \text{tr}(g^{-1} \cdot R), \quad K = \det(g^{-1} \cdot R). \quad (241)$$

Adopting the Cartesian representation (233a), we have

$$\mathbf{F}_{xx} = \begin{pmatrix} 0 \\ 0 \\ f_{xx} \end{pmatrix}, \quad \mathbf{F}_{xy} = \mathbf{F}_{yx} = \begin{pmatrix} 0 \\ 0 \\ f_{xy} \end{pmatrix}, \quad \mathbf{F}_{yy} = \begin{pmatrix} 0 \\ 0 \\ f_{yy} \end{pmatrix} \quad (242a)$$

yielding the curvature tensor

$$(R_{ij}) = \begin{pmatrix} \mathbf{N} \cdot \mathbf{F}_{xx} & \mathbf{N} \cdot \mathbf{F}_{xy} \\ \mathbf{N} \cdot \mathbf{F}_{yx} & \mathbf{N} \cdot \mathbf{F}_{yy} \end{pmatrix} = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \quad (242b)$$

Denoting the eigenvalues of the matrix $g^{-1} \cdot R$ by κ_1 and κ_2 , we obtain for the mean curvature

$$H = \frac{1}{2} (\kappa_1 + \kappa_2) = \frac{(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy}}{2(1 + f_x^2 + f_y^2)^{3/2}} \quad (243)$$

and for the Gauss curvature

$$K = \kappa_1 \cdot \kappa_2 = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}. \quad (244)$$

An important result that relates curvature and topology is the Gauss-Bonnet theorem, which states that any compact two-dimensional Riemannian manifold M with smooth boundary ∂M , Gauss curvature K and geodesic curvature k_g of ∂M satisfies the integral equation

$$\int_M K \, dA + \oint_{\partial M} k_g \, ds = 2\pi \chi(M). \quad (245)$$

Here, dA is the area element on M , ds the line element along ∂M , and $\chi(M)$ the Euler characteristic of M . The latter is given by $\chi(M) = 2 - 2g$, where g is the *genus* (number of handles) of M . For example, the 2-sphere $M = \mathbb{S}^2$ has $g = 0$ handles and hence $\chi(\mathbb{S}^2) = 2$, whereas a two-dimensional torus $M = \mathbb{T}^2$ has $g = 1$ handle and therefore $\chi(\mathbb{T}^2) = 0$.

Equation (245) implies that, for any closed surface, the integral over K is always a constant. That is, for closed membranes, the first integral in Eq. (245) represents just a trivial (constant) energetic contribution.

10.3 Minimal surfaces

Minimal surfaces are surfaces that minimize the area within a given contour ∂M ,

$$A(M|\partial M) = \int_M dA = \min! \quad (246)$$

Assuming a Cartesian parameterization $z = f(x, y)$ and abbreviating $f_i = \partial_i f$ as before, we have

$$dA = \sqrt{|g|} dx dy = \sqrt{1 + f_x^2 + f_y^2} dx dy =: \mathcal{L} dx dy, \quad (247)$$

and the minimum condition (246) can be expressed in terms of the Euler-Lagrange equations

$$0 = \frac{\delta A}{\delta f} = -\partial_i \frac{\partial \mathcal{L}}{\partial f_i}. \quad (248)$$

Inserting the Lagrangian $\mathcal{L} = \sqrt{|g|}$, one finds

$$0 = - \left[\partial_x \left(\frac{f_x}{\sqrt{1 + f_x^2 + f_y^2}} \right) + \partial_y \left(\frac{f_y}{\sqrt{1 + f_x^2 + f_y^2}} \right) \right] \quad (249)$$

which may be recast in the form

$$0 = \frac{(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy}}{(1 + f_x^2 + f_y^2)^{3/2}} = -2H. \quad (250)$$

Thus, minimal surfaces satisfy

$$H = 0 \quad \Leftrightarrow \quad \kappa_1 = -\kappa_2, \quad (251)$$

implying that each point of a minimal surface is a saddle point.

10.4 Helfrich's model

Assuming that lipid bilayer membranes can be viewed as two-dimensional surfaces, Helfrich proposed in 1973 the following geometric curvature energy per unit area for a closed membrane

$$\epsilon = \frac{k_c}{2}(2H - c_0)^2 + k_G K, \quad (252)$$

where constants k_c , k_G are bending rigidities and c_0 is the spontaneous curvature of the membrane. The full free energy for a closed membrane can then be written as

$$E_c = \int dA \epsilon + \sigma \int dA + \Delta p \int dV, \quad (253)$$

where σ is the surface tension and Δp the osmotic pressure (outer pressure minus inner pressure). Minimizing F with respect to the surface shape, one finds after some heroic manipulations the shape equation¹⁰

$$\Delta p - 2\sigma H + k_c(2H - c_0)(2H^2 + c_0 H - 2K) + k_c \nabla^2(2H - c_0) = 0, \quad (254)$$

¹⁰The full derivation can be found in Chapter 3 of Z.-C. Ou-Yang, *Geometric Methods in the Elastic Theory of Membranes in Liquid Crystal Phases*(World Scientific,Singapore, 1999).

where ∇^2 is the Laplace-Beltrami operator on the surface. The derivation of Eq. (254) uses our earlier result

$$\frac{\delta A}{\delta f} = -2H, \quad (255)$$

and the fact that the volume integral may be rewritten as¹¹

$$V = \int dV = \int dA \frac{1}{3} \mathbf{F} \cdot \mathbf{N}, \quad (256)$$

which gives

$$\frac{\delta V}{\delta f} = 1, \quad (257)$$

corresponding to the first term on the rhs. of Eq. (254).

For open membranes with boundary ∂M , there is no volume constraint and a plausible energy functional reads

$$E_o = \int dA \epsilon + \sigma \int dA + \gamma \oint_{\partial M} ds, \quad (258)$$

where γ is the line tension of the boundary. In this case, variation yields not only the corresponding shape equation but also a non-trivial set of boundary conditions.

¹¹Here, we made use of the volume formula $dV = \frac{1}{3}h dA$ for a cone or pyramid of height $h = \mathbf{F} \cdot \mathbf{N}$.

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