24 Solitons

In the previous section, we considered *dispersive* waves characterized by time-dependent change (e.g., spreading) of the wave form. Now, we will study another interesting class of *non-dispersive* waves called *solitons*.

24.1 History

The occurrence of solitons were first reported by the Scottish engineer and naval architect John Scott Russell, who described his observation as follows:

“I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped –not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour [14km/h], preserving its original figure some thirty feet [9m] long and a foot to a foot and a half [30-45cm] in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.”

Intrigued by his observation, Russell built an smaller-cases channel in his backyard and performed the first systematic studies of solitary water waves.

Another class of solitons was discovered by Enrico Fermi, John Pasta, Stanislaw Ulam (FPU) and Mary Tsingou in 1953. FPU had formulated as simple nonlinear chain model to understand better the thermalization processes (dispersion of energy) in solids. Their model was implemented numerically by Tsingou in what can be considered the first application of computers tin physics. Remarkably, instead of showing the expected thermalization, the numerical results predicted the existence long-lived non-dispersive excitations, illustrating a previously unknown transport mode.

Since then, solutions have been intensely studied as models of elementary particles and they have also been utilized in electronics and fibre optics.

24.2 Korteweg-de Vries (KdV) equation

A continuum model for solitary waves in water was first introduced by Boussinesq in 1871. The theory was developed further by Lord Rayleigh in 1876 and by Korteweg and de Vries (KdV) in 1895. According to their theory, the spatio-temporal evolution of weakly nonlinear shallow water waves is described by

$$\partial_t h = \frac{3}{2}\sqrt{\frac{g}{l}}\,\partial_x\left(\frac{h^2}{2}+\frac{2}{3}h+\frac{\sigma}{3}\partial_x^2 h\right) \tag{573}$$

where $h(t,x)$ is the surface profile of the wave and

$$\sigma = \frac{\ell^3}{3} - \frac{\gamma\ell}{\rho g} \tag{574}$$

with ℓ the channel height, γ the surface tension, g the gravitational acceleration and ρ the mass density.

Mathematical studies of the KdV-solitons typically focus on the rescaled version

$$\partial_t\phi + \partial_x^3\phi + 6\phi\partial_x\phi = 0. \tag{575}$$

To find an exact solution, we make the ansatz

$$\phi(t,x) = f(x-ct-a) = f(X). \tag{576}$$

Insertion then gives

$$-cf' + f''' + 3(f^2)' = 0. \tag{577}$$

where $f' = df/dX$. We next integrate once to obtain

$$-cf + f'' + 3f^2 = A. \tag{578}$$

This equation can be rewritten in the Newtonian form

$$\frac{d^2 f}{dX^2} = -\frac{d\Phi}{df} \tag{579a}$$

with potential

$$V(f) = -Af - \frac{c}{2}f^2 + f^3. \tag{579b}$$

This equation possesses an explicit solution with $f(X) = 0$ as $X \to \pm\infty$, given by

$$\phi(t, x) = \frac{c}{2}\text{sech}^2\left[\frac{\sqrt{c}}{2}(x - ct - a)\right]. \tag{580}$$

This solution describes a right-moving solitary wave-front of speed c.

In 1965, Zabesky and Kruskal showed how one can derive this equation in the continuum limit from the FPU model. They argued that two KdV solitons could collide and penetrate each other without exchanging energy. This explains intuitively why the FPU chain model does not lead to thermalization. Miura *et al.*[28] provided a general rigorous argument by showing that the KdV equation possesses an infinite number of non-trivial integrals of motion.

[28] Journal of Mathematical Physics 9:1204-1209, 1968