15 The coffee cup

Let's try and apply our knowledge of fluid dynamics to a real observation, to test whether the theory actually works. We shall consider a question you will encounter in the last problem set[21]: how long does it take a cup of coffee (or glass of water) to spin down if you start by stirring it vigorously? To proceed, we need a model of a coffee cup. For mathematical simplicity, let's just take it to be an infinite cylinder occupying $r \leq R$. Suppose that at $t = 0$ the fluid and cylinder are spinning at an angular frequency ω, and then the cylinder is suddenly brought to rest. Assuming constant density, we expect the solution to be cylindrically symmetric

$$p(\boldsymbol{x}, t) = p(r, t) , \qquad \boldsymbol{u}(\boldsymbol{x}, t) = u_\phi(r, t)\boldsymbol{e}_\phi, \tag{398}$$

with only a component of velocity in the angular direction $\boldsymbol{e}_\phi$. This component will only depend on r, not the angular coordinate or the distance along the axis of the cylinder (because the cylinder is assumed to be infinite).

We shall just plug the assumed functional form into the Navier-Stokes equations and see what comes out. Let's put our ansatz $p = p(r, t)$ and $\boldsymbol{u} = (0, u_\phi(r, t), 0)$, which satisfies $\nabla \cdot \boldsymbol{u} = 0$, into the cylindrical Navier-Stokes equations. The radial equation for $\boldsymbol{e}_r$-component becomes

$$\frac{u_\phi^2}{r} = \frac{\partial p}{\partial r}. \tag{399a}$$

Physically this represents the balance between pressure and centrifugal force. The angular equation to be satisfied by the $\boldsymbol{e}_\phi$-component is

$$\frac{\partial u_\phi}{\partial t} = \nu \left(\frac{\partial^2 u_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r^2} \right), \tag{399b}$$

and the vertical equation is

$$0 = \frac{1}{\rho}\frac{\partial p}{\partial z}. \tag{399c}$$

The last equation of these three is directly satisfied by our solution ansatz, and the first equation can be used to compute p by simple integration over r once we have found u_ϕ.

We want to solve these equation (399a) and (399b) with the initial condition

$$u_\phi(r, 0) = \omega r, \tag{400a}$$

and the boundary conditions that

$$u_\phi(0, t) = 0 , \qquad u_\phi(R, t) = 0 \qquad\qquad \forall t > 0. \tag{400b}$$

This is done using *separation of variables*. Since the lhs. of Eq. (399b) features a first-order time derivative, which usually suggests exponential growth or damping, let's guess a solution of the form

$$u_\phi = e^{-k^2 t}F(r). \tag{401}$$

Putting this into the governing equation (399b) gives the ODE

$$-k^2 F = \nu \left(F'' + \frac{F'}{r} - \frac{F}{r^2} \right) \tag{402}$$

[21] See Acheson, *Elementary Fluid Dynamics*, pp. 42-46

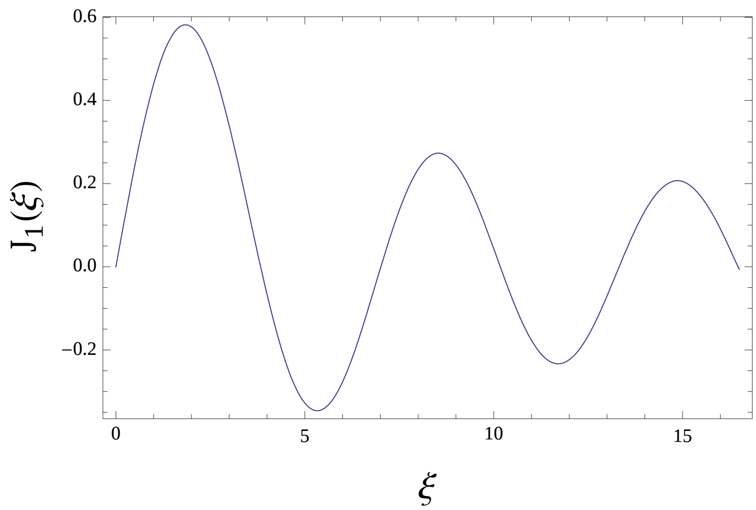

Figure 3: The Bessel function of the first order $J_1(\xi)$.

This equation looks complicated. However, note that if the factors of r weren't in this equation we would declare victory. The equation would just be $F'' + k^2/\nu F = 0$, which has solutions that are sines and cosines. The general solution would be $A\sin(k/\sqrt{\nu}r) + B\cos(k/\sqrt{\nu}r)$. We would then proceed by requiring that (a) the boundary conditions were satisfied, and (b) the initial conditions were satisfied.

We rewrite the above equation as

$$r^2F'' + rF' + \left(\frac{k^2}{\nu}r^2 - 1\right)F = 0, \tag{403}$$

and make a change of variable,

$$\xi = kr/\sqrt{\nu}.$$

The equation becomes

$$\xi^2F'' + \xi F' + \left(\xi^2 - 1\right)F = 0. \tag{404}$$

Even with the factors of ξ included, this problem is not more conceptually difficult, though it does require knowing solutions to the equation. It turns out that the solutions are called *Bessel Functions*. You should think of them as more complicated versions of sines and cosines. There exists a closed form of the solutions in terms of elementary functions. However, people usually denote the solution to the Eq. (404) as $J_1(\xi)$, named the *Bessel function of first order*[22]. This function is plotted in Fig. 3, satisfies the inner boundary conditions $J_1(0) = 0$. For more information, see for example the book *Elementary Applied Partial Differential Equations*, by Haberman (pp. 218-224). Now let's satisfy the boundary

[22] Bessel functions $J_\alpha(x)$ of order α are solutions of

$$x^2J'' + xJ' + \left(x^2 - \alpha\right)J = 0$$

condition $u_\phi(R,t) = 0$. Since we have that

$$u_\phi = AJ_1(\xi) = AJ_1(kr/\sqrt{\nu}), \tag{405}$$

this implies that $AJ_1(kR/\sqrt{\nu}) = 0$. We can't have $A = 0$ since then we would have nothing left. Thus it must be that $J_1(kR/\sqrt{\nu}) = 0$. In other words, $kR/\sqrt{\nu} = \lambda_n$, where λ_n is the n^{th} zero of J_1 (morally, J_1 is very much like a sine function, and so has a countably infinite number of zeros.) Our solution is therefore

$$u_\phi(r,t) = \sum_{n=1}^{\infty} A_n e^{-\nu\lambda_n^2 t/R^2} J_1(\lambda_n r/R). \tag{406}$$

To determine the A_n's we require that the initial conditions are satisfied. The initial condition is that

$$u_\phi(r) = \omega r. \tag{407}$$

Again, we now think about what we would do if the above sum had sines and cosines instead of J_1's. We would simply multiply by sine and integrate over a wavelength. Here, we do the same thing. We multiply by $rJ_1(\lambda_m r/R)$ and integrate from 0 to R. This gives the formula

$$A_n \int_0^R rJ_1(\lambda_n r/R) J_1(\lambda_m r/R)\, dr = \int_0^R \omega r^2 J_1(\lambda_m r/R) dr. \tag{408}$$

Using the identities

$$\int_0^R rJ_1(\lambda_n r/R) J_1(\lambda_m r/R)\, dr = \frac{R^2}{2} J_2(\lambda_n)^2 \delta_{nm}, \tag{409}$$

and

$$\int_0^R \omega r^2 J_1(\lambda_m r/R) dr = \frac{\omega R^3}{\lambda_m} J_2(\lambda_m), \tag{410}$$

we get

$$A_n = -\frac{2\omega R}{\lambda_n J_0(\lambda_n)}, \tag{411}$$

where we have used the identity $J_0(\lambda_n) = -J_2(\lambda_n)$. Our final solution is therefore

$$u_\phi(r,t) = -\sum_{n=1}^{\infty} \frac{2\omega R}{\lambda_n J_0(\lambda_n)} e^{-\nu\lambda_n^2 t/R^2} J_1(\lambda_n r/R). \tag{412}$$

Okay, so this is the answer. Now lets see how long it should take for the spin down to occur. Each of the terms in the sum is decreasing exponentially in time. The smallest value of λ_n decreases the slowest. It turns out that this value is $\lambda_1 = 3.83$. Thus the spin down time should be when the argument of the exponential is of order unity, or

$$t \sim \frac{R^2}{\nu\lambda_1^2}. \tag{413}$$

This is our main result, and we should test its various predictions. For example, this says that if we increase the radius of the cylinder by 4, the spin down time increases by a factor of 16. If we increase the kinematic viscosity ν by a factor of 100 (roughly the difference between water and motor oil) then it will take roughly a factor of 100 shorter to spin down. Note that for these predictions to be accurate, one must start with the *same angular velocity* for each case.

In your problem set you are asked to look at the spin down of a coffee cup. From our theory we have a rough estimate of the spin down time, which you can compare with your experiment. Do you get agreement between the two?