Assignments are to be submitted to Gradescope by 24:00.

1. Let $M \subset \ell^\infty$ be the subspace
$$M = \left\{ a = \{a_k\}_k \in \ell^\infty \mid \lim_{k\to\infty} a_k \text{ exists} \right\}.$$
   (a) Define $u : M \to \mathbb{C}$ via
$$u(a) = \lim_{k\to\infty} a_k, \quad a \in M.$$
   Prove that u is a bounded linear functional on M.

   (b) By the Hahn-Banach theorem, there exists $v \in (\ell^\infty)'$ such that $v|_M = u$. Prove that there is no $b = \{b_k\}_k \in \ell^1$ such that for all $a \in \ell^\infty$
$$v(a) = \sum_{k=1}^\infty a_k b_k.$$
   Hint: Towards a contradiction, suppose that there exists $b = \{b_k\}_k \in \ell^1$ such that for all $a \in \ell^\infty$, $v(a) = \sum_k a_k b_k$, and consider $v(e_n)$ where $e_n = \{\delta_{nk}\}_k \in \ell^\infty$.

2. Suppose that V and W are normed vector spaces and $T \in \mathcal{B}(V, W)$.

   (a) Define $T^\dagger : W' \to V'$ by
$$T^\dagger f := f \circ T, \quad f \in W'.$$
   Prove that $T^\dagger \in \mathcal{B}(W', V')$ and $\|T^\dagger\| = \|T\|$. $T^\dagger$ is called the *adjoint* or *transpose* of T.
   Hint: First show that $\|T^\dagger\| \le \|T\|$. To show the reverse inequality, let $x \in V$ with $\|x\| = 1$. A corollary of the Hahn-Banach theorem proved in lecture provides an element $f \in W'$ such that $\|f\| = 1$ and $f(Tx) = \|Tx\|$. Use this fact to show that $\|Tx\| \le \|T^\dagger\|$.

   (b) Let $1 \le p \le \infty$. For $a = \{a_k\}_k$, define the *right shift* of a to be the sequence
$$\{Ra\}_k := \{0, a_1, a_2, a_3, \ldots\}.$$
   Prove that $R \in \mathcal{B}(\ell^p, \ell^p)$ and compute $\|R\|$.

   (c) Let $1 \le p < \infty$ and $1/p + 1/q = 1$. Recall from Assignment 1 that we can identify $(\ell^p)'$ with ℓ^q via "pairing": $f \in (\ell^p)'$ if and only if there exists $b \in \ell^q$ such that for all $a \in \ell^p$
$$f(a) = \sum_{k=1}^\infty a_k b_k,$$
   and moreover, $\|f\| = \|b\|_q$. With this identification of $(\ell^p)'$ with ℓ^q, compute $\{(R^\dagger b)_k\}_k$ in terms of $b = \{b_k\}_k \in \ell^q$. For example, if $b = e_1 = \{1, 0, 0, \ldots\}$, then for all $a \in \ell^p$
$$(R^\dagger e_1)(a) := \sum_{k=1}^\infty (Ra)_k (e_1)_k = (Ra)_1 = 0 = \sum_{k=1}^\infty a_k \cdot 0,$$
   so $R^\dagger e_1 = \{0\}_k = 0 \in \ell^q$.

3. Let $E \subset \mathbb{R}$ and $x \in \mathbb{R}$. Prove that if $E + x := \{y + x \mid y \in E\}$, then

$$m^*(E + x) = m^*(E).$$

[Thus, outer measure is translation invariant.]

4. This exercise shows that every open set can be written as a countable union of open intervals. Let $U \subset \mathbb{R}$ be an nonempty open set. For $x \in U$, define

$$a_x := \inf\{a \in \mathbb{R} \mid (a, x] \subset U\}, \quad b_x := \sup\{b \in \mathbb{R} \mid [x, b) \subset U\}.$$

(a) Prove that for all $x \in U$, $(a_x, b_x) \subset U$.

(b) Prove that if $x, y \in U$ and $y \in (a_x, b_x)$ then $(a_y, b_y) = (a_x, b_x)$.

(c) Prove that

$$U = \bigcup_{q \in U \cap \mathbb{Q}} (a_q, b_q).$$

Hint: Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$: for all real numbers $c < d$ there exists $r \in \mathbb{Q}$ such that $c < r < d$.

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