[SQUEAKING]

[RUSTLING]

[CLICKING]

PROFESSOR: So let's finish our discussion of the Hahn-Banach Theorem. Let me just recall. This was the theorem that stated, that if v is norm space, m is a vector-- or is a subspace of v.

And u is a bounded linear functional on m. So it's a linear map that also satisfies for all t and m, u of t, and absolute value is-- or the modulus of u of t is less than or equal to a constant times norm of t.

Then there exists a bounded linear functional on the whole space, which extends it. So that there exists a capital U, so that when U is restricted to m this gives me little u. And it's extended continuously in the sense that it satisfies the same bound that little u does.

Now for all t and v. u of t is less than or equal to a constant times the norm of t. So last time, we went through the proof of this using Zorn's lemma and the lemma that we ran up against at the end of class.

Which, when writing my notes for the lecture, I got a little bit flippant towards the end and made a little small error. Which can be remedied if you just look at the lecture notes, which I have online. Or if you look at Richard's lecture notes just for the very end of concluding that Lemma we needed.

So minus epsilon, we proved the Hahn-Banach Theorem. You can fill in the epsilon by just looking at the notes. But the proof of the Hahn-Banach Theorem is not the point. It's not the main point. It's important to know it, or at least have seen it, but what is important is it as a tool.

And I mentioned last time, one application is to show that the dual of little l infinity is not little l1. Another application is the following is, so v is going to be a norm space.

Then for all v and t, take away 0. There exists a bounded linear functional and element of the dual space such that the norm of f equals 1. And f of v gives me the norm of v.

So there is a bounded linear functional that, when you apply it to v, you get the norm of v. And we'll use this just in a second to say something a little bit deeper about the relation between the dual and then taking the dual of that. But let's prove this simple theorem.

So first, I define. So I have a little v that's non-0 in the space, and I define a bounded linear-- or a linear map from the span of v to c by u of lambda v equals lambda times the norm of v.

Now, every element that's a scalar multiple of v can be written uniquely as this. So this function is well-defined, and it's clear that it's linear because linearity is occurring in lambda. v is fixed. Remember?

So, in particular, I get that u of t is less than or equal to u-- to the norm of t for all t in the span of this fixed vector v. And you have one-- or I should say, u of v, which, remember, is just a-- So this is equal to just 1 times the norm of v. It gives me norm of v.

And so, then, by the Hahn-Banach Theorem, there exists an f element of the dual extending u. Such that now, for all t and v, f of t and modulus is less than or equal to the norm of t.

And in particular. So since it extends, u must have equals the norm of v. And by that inequality over there, the norm of little f must be less than or equal to 1.

Well, first, let me finish with what this inequality tells you. This tells you that-- So since f of t is less than or equal to t for all t and v, this implies that the norm of f--

Which, remember, is the sup of f of t with t unit length. It must be less than or equal to 1. But I have that 1 is equal to f of b over norm of b, which must be less than or equal to the norm of f. And, therefore, the norm of f equals 1.

So f is this desired, continuous linear functional on v that, when it hits v, little v, it spits out the norm of little v.

The double dual of a space is-- if you can take the dual of a space, you can take the dual of that space. The double dual of a norm space v is, by definition, we denote it by v double prime.

This is the dual of the dual of v. So, remember, the prime is the space of bounded linear functionals on v. And so, then, v double prime is the space of bounded linear functionals on the space of bounded linear functionals of v.

Now, let me just give you an example of-- so, a quick little example of an element that you can associate in v double prime. Fix an element in capital V and define.

Now, let's call it something. Let's call it t of v. So this will be an element of the dual space of v prime in the end. But t sub v by--

So this thing has to eat an element of the dull space and spit out a number. Now, think about the two pieces of data I have here. I have some fixed little v and capital V, and I would like to use that to define some bounded linear functional on capital V prime.

Now, capital V prime eats elements of capital V. So I can define a map from V prime to C by that. Then I claim that T of V is an element of the double dual. So the dual space of V prime.

And why is that? So first off, it has to be linear in the argument. So little v prime is in capital V prime. It has to be linear in the argument V prime. And this is clear. So remember, little v is fixed. So if I take a linear combination of two elements of capital V prime, then this expression is clearly linear in V prime. So T sub V is linear clearly.

Now we just need to check that it's bounded. And if I take-- so T sub v is linear. This is a check. Just how we talk through it. And T sub v is bounded because if we take T sub v of v prime, if we take its modulus, this is by definition equal to v prime of v.

Now, v prime is a bounded linear functional on v. So this is less than or equal to the norm of v prime times the norm of v. And in fact, let me write it this way. Norm of v, v prime.

So we've shown that this linear operator from v prime to C in modulus is bounded by this constant times the norm of v prime. And therefore, we conclude that T of v-- or T sub v is in the dual of V prime, the double dual.

And the norm of this as an operator going from v prime to C, which we've just shown, is bounded by this constant. Remember, the operator norm is the best constant that appears here. And therefore, this is less than or equal to that.

So we've shown that every element of v, we can associate an element of the double dual, the dual space of the dual space, via this relation. So an element of v can be defined as an element of v double prime by letting it act on v prime by this formula here.

But we can say a little bit more. And let me just introduce some terminology. Let v be in capital V. Or, no, I'm not quite there. If V and W are normed spaces, then we say that a bounded linear operator from V to W is isometric.

So isometric meaning doesn't change length, doesn't change distances. If for all v and capital V the length of T of V is equal to the length of V.

And now, the next theorem is that this map, this relation I've given between elements of v and the double dual is, in fact isometric. So let v be in V. And as before, define T sub v as the map from v prime to C via T sub v needs to be elements of the dual. So you can spit out numbers so it's defined in this way.

Then the map T from V to V double prime, where this map T is taking V to T little v is isometric. So it's a bounded linear operator from V to its double dual. And the length of a vector in v is same as the length of its image in the double dual. Which is the length in the double dual is defined in terms of the operator norm.

So we've done basically all of the work when I was discussing the example. So we've shown already that the map v to T of v, this is a bounded linear operator from v to v double prime. So we showed it's bounded, namely that the image is bounded by-- or the norm of the image is bounded by the norm of v.

But it's also clear that this is linear in little v. So a minute ago, we showed it's linear in v prime. But it's also linear in v. Because for each fixed v prime, this is linear in v. So this is a bounded-- this map from v taking you to this element in T of v, this is a bounded linear operator from v to v double prime.

So what's left is to show that it's isometric. So we've shown this and that the norm of the image in the double dual is less than or equal to the norm of v. Now to show it's isometric, we must show that the norm of v equals T of v.

So by the theorem we just proved a minute ago, so first off, so as in the statement of the theorem, let me denote this map by this is T. So we've shown that the norm of T is less than or equal to 1. Because we've shown that the norm of the image is always less than or equal to the norm of the input.

And now to show that it's isometric we just need to show the norm of T equals 1. So let v be in v non-zero. with norm v equal to 1.

Well, so I just said something really stupid a minute ago. Let me go back to what I had written down there a minute ago. So not that-- so remember, we're trying to show not that the norm is one but that the norm of the image is equal to the norm of the input. I got backwards there for a minute, sorry.

And the norm of the image is less than or equal to the norm of the input. So now we just want to show the reverse inequality for all little v in capital V. Now we show all v in capital V, T of v is equal to the length of v.

And to do that, we use this theorem we just proved a minute ago. So this is clear if v is 0. So suppose v is non-zero, then there exists an element of the dual space by the previous theorem such that the norm of f equals 1 and f of v equals the norm of v.

Then the norm of v is equal to f of v. And I can even put modulus on that. And this is less than or equal to though, thinking of this as a-- so now thinking of this as an operator from bounded linear functional, so [INAUDIBLE], so as an element of the dual this is less than or equal to the norm of T sub v times the norm of f.

And now the norm of f is 1. So this is equal to the norm of T sub v. And therefore, v, the norm of v, is less than or equal to the norm of T sub v. And since I already had the reverse inequality, I conclude that the norm of T sub v equals the norm of v.

And thus, this map going from v to v, to the double dual of v described this way is, in fact, isometric. It's a bounded linear operator that preserves links.

Now we have a special name. So first off, it should be clear that isometric bounded linear operators are one to one. So for something to be one to one for a linear operator, that means the only thing that gets sent to 0 is 0. And from this equality here we have that if this equals 0, then the vector had to be 0.

So isometric bounded linear operators are always one to one. So what this theorem tells you is that this map defined in this way gives you a isometric injection from v to v double prime, from v to its double dual.

So I have this map that goes into the double dual that's isometric, meaning it doesn't change distances. Is it onto? Is the double dual always equal to the original space itself? So we have a name for spaces that satisfy that.

So a Banach space V is reflexive if using-- if this map is onto. If V equals v double prime, in the sense that this map taking element of capital V to an element of the double dual that, as we defined earlier, is onto.

Now for example, and you can check that. This may seem like abstract stuff. But for little lp spaces, this letting the dual space eat a vector or whatever, remember, we identify the dual space of little lp with little lq, where 1 over p plus 1 over q equals 1.

So little lp is reflexive for p between 1 and infinity because the dual of little lp is going to be little lq, where 1 over p plus 1 over q equals 1. And the dual of 1 over-- of little lq is going to be a little lp as long as q is between 1 and infinity.

There is that one case where the dual is not given by what you think it should be. So little l1 is not reflexive. Since if I take a look at the dual of little l1, so this should be 1, this is equal to l infinity.

And as you'll show in the homework, the dual of little l infinity is not equal to little l1. So the dual of the dual is not going to give you back the space you started with.

And I don't know if I'm going to put this in the assignment or eventually on a midterm. The space C0, which was the space of all sequences converging to 0, this is not reflexive either. The reason is because you can, in fact, identify the dual of C0 with little l1. And the dual of little l1 is little L infinity. Which does not equal C0.

So this also, you can see how the space is a subspace of the double dual. The space of all sequences converging to 0, that's a subspace of the space of all bounded sequences. So you see this, I shouldn't say by hand, but OK, by hand, that the original space-- forget this prime-- is a subspace of the double dual which in this case is little infinity.

So that concludes the general stuff about Banach spaces for now. And we're going to turn to Lebesgue measure and integration. Because so we've been talking about little lp spaces. These are spaces of sequences.

And so you might think, well, maybe we can define big LP spaces to be, let's say, Riemann integral functions whose p'th power is integrable or something like that. Just how these are p'th power summable.

So moving on now to Lebesgue measure and integration. Missing a letter there.

Now, why the Lebesgue measure and integration, why not just stick to Riemann integration? There's a couple of reasons. One is Lebesgue integration has much better convergence theorems.

So you only have really one convergence theorem for Riemann integration, which is the uniform limit of Riemann integrable functions is Riemann integrable. And the integral of the limits is the limit of the integrals.

But there are much better limit theorems. And in the sense that they're more useful, you can use them more often, and therefore prove better things. And maybe you're just like, OK, so what?

But there's even bigger reasons why you consider Lebesgue measure and integration. And this is because that if you look at the space of Riemann integrable functions on a, b, say, 0, 1, say, let's make it concrete, then let me in fact write this down. L1-- and I'm going to put an R here.

This is a set of all f from 0, 1 C such that f is Riemann integrable on 0, 1. Now in 18.100 and 100A, these are usually real-valued. But when I mean Riemann integrable and its complex valued, I just mean the real part and the imaginary part are Riemann integrable.

So you don't need to know anything fancy. And if I define a norm on capital L1 by-- so I have this space I have this norm.

Now, it's not quite a norm. Because you can have Riemann integrable functions which are not 0 everywhere, whose integral is 0. If you have a function that's 0 except for at one point, and at that one point it's 5, well, the function's non-zero, but the integral is 0. So this is a semi-norm. But let's imagine it's a norm.

So the problem is, is that even if I mod out by those things that give me a semi-norm equal to 0 so that I get an actual norm space, but ignore that for a minute. Let's just imagine that this is an actual honest to God norm. Then this thing ends up being this is not a Banach space.

So one reason to consider more general integration is because Riemann integration, or at least only restricting to those functions which are Riemann integrable does not give you a Banach space. This is not-- this is a norm space modulo this little bit that there are functions that have norm 0 that aren't exactly identically 0.

But it's not a Banach space. It's not complete. And in functional analysis, we're interested in spaces that are complete. And not just in functional analysis for the abstract love of it, but in problems where, as I said at the beginning of this course, where you have differentiable equations or some functional defined on some space functions you want to deal with complete spaces.

And a lot of functionals are defined on not just L1 but say LP, where now I replace this by a p'th power. So in order to be able to say certain things exist or get to the heart of the subject, I need these spaces to be complete. And if I just restrict to Riemann integrable functions, it's not complete. So we have to do Lebesgue integration.

Now, what we'll find out is that-- and this is how it's done in the original notes for this class, is that this is not a Banach space. But I can take its completion. Just like the set of rational numbers is not complete, I can take it's what's called completion and get the set of real numbers.

I could take this space, take its completion, and what I get is a abstract space, which I can actually identify with the space of Lebesgue integrable functions. That's how it's done in the notes.

But I don't want to do it that way. Instead, we're just going to build up Lebesgue measure and integration from the ground up. And we will see at one point that these functions are in fact dense in the space of Lebesgue integrable functions. And therefore, we conclude that the abstract completion-- if you haven't covered completions, that's fine-- of this space is, in fact, the integrable-- the space of Lebesgue integrable functions.

So summing it up is that we have to do Lebesgue measure and integration because these spaces are not complete. And we have better convergence theorems in these spaces. But it just takes a little bit of effort to learn. Not much, but I mean, it's quite intuitive once you start seeing the arguments and going through it.

So we're going to be defining a new notion of integration that's more general than Riemann integration. And so, integration should somehow be the theory of area underneath the curve.

So the simplest type of functions you should consider first is, if I have a subset E, maybe it's crazy, or maybe it's simple. And I have a function, which is just a 1 when I'm on it and 0 otherwise. So I'll denote this function by 1 with an E beneath it.

Then in some sense, I want to be able to integrate this function. So 1E is the function that's 1 if x is in E and 0 if x is not in E. So if E is just the interval a, b, then I just have the function that 0 outside of a, b and 1 over a, b.

So the question is, how do we integrate such functions? Or our first task is to define the integral for these types of functions. Well, since the integral should be a theory of the area underneath the curve, for example, if E equals-- so this is all just discussion right now, I should add. So this is motivation. In a minute I'll get to definitions and theorems. But this is just a discussion.

If E equals a, b so that I'm just-- I have a, b, 1, 0 outside, then the integral of this guy in whatever sense this integral is, this should be at least the area underneath this curve. So it should be b minus a. Which is the length of the interval a, b.

So and therefore, if we have a general set E, we should expect that our integral over the function, which is 1 on E and 0 off it, should somehow-- this should be the length of E. But length is not a very good word because length applies to an interval a, b because there's a start and there's a stop. And everything in between is in the set.

So rather than write the length of E, I'll write m of E. And m of E being the what I'll say is a measure of E. A measure of how much of E there is.

And so, our first task, if we're hoping to develop a notion of integration more general than Riemann, in which the resulting spaces are a Banach space, that's really the goal in the end we should start by defining what does this mean? What is a measure of a set? Which sets are we measuring?

So our task right now is define before we even get to integration, we should be able to integrate the simplest types of functions. And this requires us to be able to define the measure of subsets of R. And this is Lebesgue measure that we'll be developing.

So what are some properties that we want that a reasonable measure of sets to have? So the first one we would like is, we should be able to measure everything.

What's awful about Riemann integration is we can't really integrate any function. But we can't even integrate this function when E is, say, the rational numbers between 0 and 1. When I have the function which is 1 on the rationals and 0 off of that, that's not Riemann integrable.

So I would like to be able to measure any subset of real numbers. So I would like to be able to define the measure of any subset of real numbers. And the second property I would like is kind of a sanity check, that if I is an interval, then the measure of E should be the length of I.

And by I, I mean half open, half closed. I guess, a half open is half closed. But closed, open interval, the measure of that should not care about missing the two endpoints. And I should just get out the length of that interval b minus a.

A third property is a measure of the whole should be the sum of the measures of its parts. If I have some set which can be written as a union of disjoint chunks, then the measure of that whole set should be the sum of the measure of the individual chunks.

So that's stated as if En is a collection of disjoint sets. So think of these as making up a bigger set. And this is, I should say, a countable collection of disjoint sets, then I would like for the measure of their union to be the sum of the measures. This is a reasonable thing to ask for. The measure of the whole is the sum of the measure of the parts.

And in the last one, which is kind of specific to how we view R is, if I take a set here next to where I'm standing, and then I take that set, don't do anything to it, I just walk it over here and take it's measure, that measure, the measure of this set which I've now walked over here should be the same as the measure of that set over when I was standing over here.

So we should like m to be translation invariant. Meaning if E is in R and x is an element of r, then the measure of the set x plus E, which means the measure of the set of elements x plus y, where y is in E. So just take E and shift it by x. This should be the same as the measure of the original set E.

So unfortunately, this is impossible to have a function which is defined on every subset of real numbers satisfying these three properties. So that's very unfortunate.

Such a function m going from the power set of R, meaning the set of all subsets of R, and of course this thing should be non-negative since it's a measure of how much is there, does not exist.

Meaning what? If you assume all of these four properties, you'll be able to come up with a set which has finite measure and then also simultaneously has infinite measure. So it's impossible to have a function which is defined for every subset of real numbers satisfying the conditions two, three, and four. It's just not logically-- it's just not possible.

So that's unfortunate. But what we can do is drop the assumption that measure is defined for every subset of real numbers and focus on trying to find a set function-- I mean, defined on subsets of real numbers. So that's why I'm calling it a set function-- m which satisfies two, three and four on a big collection of sets, that's defined on a large collection of sets.

And these sets for which such a measure will be defined on satisfying two, three, and four will be quite large in the end. This is the set of Lebesgue measurable sets. And m is the Lebesgue measure. And so, our goal is to construct the Lebesgue measure and Lebesgue integrable sets.

So this is our task. Or I guess we had that task before. So what is the plan now? Construct an m defined for a large class-- so defined for many different sets but not necessarily-- but not every set.

And these sets will be, we will call Lebesgue measurable sets such that the conditions two through four hold. That if I have an interval, the measure of that interval-- so first off, this class of sets should contain all intervals. And the measure of that interval gives me the length of the interval.

And if I have a countable collection of sets for which I can measure, then their union is measurable. And the measure of that disjoint union is equal to the sum of the measures. And then it's also translation-invariant.

So that's the plan. So it won't be defined on for every set. But it will be defined for a large class of sets which contain most reasonable sets. And how we're going to do this is due to [INAUDIBLE].

And we'll go about it like this. We'll first-- so how we'll construct this, we'll first construct a function m star, which is defined for every subset of real numbers. This we'll call the outer measure, which satisfies two, namely that the m star of an interval is the length of the interval 3. And I shouldn't say 3, but and almost 3.

Then we restrict, I should say, satisfying two, almost three, and four. Then we restrict m star to well-behaved subsets of R. These will be the class of the big Lebesgue sets. And m will just be m star restricted to these subsets.

So this is the plan of this chapter, this part of the course. Again, this is all discussion. So our first topic will be m star, which is called outer measure.

So this was all game plan discussion. If you didn't follow everything I just said, that's fine. You could start listening now. Because then I'm just going to start defining things and proving theorems about them. But you should know the path that we're on so that you don't lose the forest through the trees or something like that. I think it goes something like that.

Anyways, so outer measure-- so for an interval-- so let me just note a little notation, as I used a minute ago. If I is an interval, meaning open closed, half open, infinite, little l of I denotes its length. If it's unbounded, then this just is a stand-in for infinity.

And the length of the interval-- so the length of a, b both including a, b or not, or over 1 and maybe not the other, is equal to b minus a regardless if it's an open, closed, or half open interval.

So for a subset of real numbers, we define its outer measure-- we define the outer measure of a. This is m star of a. This is equal to the infimum of numbers sum of lengths of I sub n.

And what are the I sub n's? This is equal where I sub n is a countable collection of open intervals such that a is contained in their union.

So how I compute the outer measure, so I can cover any set-- any subset of the real numbers by a union of open intervals. And so then, the outer measure is then defined as the infimum of the sum of the lengths of these open intervals.

So you have your set. You cover it by open intervals. You sum up the length of the open intervals. This gives you some rough approximation to the size of a. And now if you make those intervals small, and smaller, and smaller, then you should be picking up more detailed information. And so, the infimum of some of the length of these intervals that are covering a, this gives you the outer measure.

So let me give you the stupidest example ever, which is the outer measure of a point. So let's say, I don't know, the set containing just 1. So I claim that-- or the set containing 0. The outer measure of this is just 0.

So the set with just one point should-- it doesn't fill anything. So it shouldn't have any measure, it shouldn't have a positive measure. At least if we're wanting to have measure be related to length.

Now, why is this? Well, let epsilon be positive. What I'm going to show is that the outer measure of this set is less than epsilon. And since this is always a non-negative number, it's the infimum over a subset of positive numbers. So therefore, that infimum is always non-negative.

So I should note that this is always bigger than or equal to 0. So I'm going to show that the outer measure is less than epsilon. Then the set containing only 0 is contained in the open interval epsilon over 2-- minus epsilon over 2 to epsilon over 2.

And therefore, the outer measure of 0, which is the infimum of the sum of links over all collection of intervals covering a, this should be less than or equal to if I just take one of these. So length of, which equals epsilon.

Now, the measure of this is always less than or equal to epsilon for all epsilon positive. So then I can take-- send epsilon to 0. And I get that this measure-- outer measure is 0. So again, the outer measure is the infimum of the sum of lengths, of intervals, of open intervals, where these open intervals cover a.

Simplest one is the set containing a single point. It has measure 0. Now I bet if you sit and think for a little bit, you can then prove that the outer measure of a set containing finitely many points, this is also 0.

And if you work a little bit harder, you can even prove that the measure of a countable set is also 0 just based on what we've done here, just based on-- you know what? Let's do that. That's a fun exercise.

So is countable, then the measure of a equals 0. So for example, the rational numbers are countable. There's a lot of rational numbers throughout R. But they have measure 0. They don't fill anything in a certain sense. Not feel as in F-E-E-L, fill, as an F-I-L-L.

So what's the proof of this? So if a is countable, then I can list the elements of a. I mean, a being countable means there's a bijection from a to the natural numbers. And I'll do the countably infinite one and leave the finite case to you.

So that means there exists a bijection from a to the natural numbers. But that just means that I can list the elements of a, a1, a2, a3, a4, and so on. You can write it as-- OK.

Now, I am, just like I did a minute ago, I'm going to show that the outer measure of this set is less than or equal to epsilon, where epsilon is an arbitrary positive number. And therefore, the outer measure has to be 0. Let epsilon be positive.

We will show the outer measure of a, which is, again, a non-negative number is less than or equal to epsilon. And then, any number that's less than or equal to an arbitrarily small number has to be 0.

For each n natural number, let I n be the interval that takes the form epsilon over 2n plus 1. I should write it this-- a sub n minus epsilon over 2n plus 1, a sub n plus epsilon over 2n plus 1.

So I sub n is an open interval that contains a sub n for each n. And so, a is contained in this countable union, or this countable union of open intervals.

And since the outer measure is the infimum of the sum of lengths of open intervals covering a, this implies that the measure, the outer measure of a, which again is the infimum, so it's smaller than sum of lengths of open intervals covering a, this is less than or equal to sum n equals 1 to infinity of the length of I n, which equals sum from n equals 1 to infinity of now the length of each of these intervals is epsilon over 2 to the n.

And the sum from n equals 1 to infinity of epsilon over 2 to the n, this is just epsilon. And therefore, the outer measure is less than epsilon. Since epsilon was arbitrary and the outer measure is non-negative, that we conclude that the outer measure has to be 0.

I mean, this argument, we can generalize this argument a little bit to prove something that I'm getting a little out of-- what I'm about to do is slightly out of order. But we can generalize the argument we just gave. And I want to do it now since it's there and we just did it, to prove that outer measure satisfies something that's almost like three.

So and it's kind a generalization of this argument. Except for, this argument we were able to give an explicit I sub n. So we have the following theorem that, well, first off let me state that a very easy property of outer measure is that if I have two subsets of real numbers, and a is contained in b, then the outer measure of a is less than or equal to the outer measure of b.

Because this just follows from the definition. Any covering of b by open intervals is a covering of a by open intervals. And therefore, the infimum has to be less than or equal to the sum of those lengths.

Or I should say, then the outer measure has to be less than or equal to the sum of those links. And thus, the infimum over all of those intervals covering b, which is the outer measure, has to be bigger than or equal to the outer measure of a.

So just think about it for a minute. But this should be pretty much clear from the definition.

So the next theorem is that we have something that's like three. Let a and A n be a collection-- be a countable collection, I should say. So here n is a natural number, collection of subsets of R.

Then the outer measure of their union-- so this is just arbitrary subsets. I mean, they don't have to be pairwise disjoint like one and three. But so, just an arbitrary collection. If I take the outer measure of the union, this is less than or equal to the sum of the outer measures.

So in particular, if this collection is a collection of pairwise disjoint sets, then the measure of the hole is less than or equal-- less than or equal to the sum of the measure of the parts, which is half of the three that we want. So we're getting there. But anyways, so let's prove this.

First off, if there exists an n such that it has infinite outer measure, just meaning that that subset that I'm taking the infimum over is unbounded, or-- no, no, no. That's not what I mean.

But if that infimum is infinite, meaning I can't cover the set by a collection of intervals with sum of links as a finite number, or the sum is infinite, meaning it's not convergent, then this inequality is true.

I mean, then I'm just saying something is less than or equal to infinity. Or now I'm treating infinity like an element of the extended real numbers.

So if one of the measures is infinite, or if this series diverges and I just get an infinite number, then this inequality holds trivially. So let's just consider the case when-- so we just need to consider the case when all of the measures are finite and the sum of the measures is finite.

So suppose while n be an outer measure of and the outer measure-- the sum of the outer measures converges. So it's finite.

So this is an argument you have to get used to, that instead of proving the inequality that you want to prove, you prove this inequality plus epsilon, where epsilon is an arbitrary thing-- arbitrary positive number, just to give yourself a little room. And then if you're able to prove the inequality you want, plus epsilon, where epsilon is an arbitrary positive number, then the inequality holds by letting epsilon go to 0.

So that's the goal is we're going to prove that inequality with an epsilon on the right-hand side. So plus an epsilon. Let epsilon be positive. For each n let collection I n sub k, kn natural number, be a collection of open intervals covering a sub n-- so I sub n sub k.

And so remember, the outer measure is an infimum. So if I go above this infimum a little bit, I can always find a collection of open intervals covering the set whose sum of links is less than that infimum plus the small number. So and sum of k equals 1 to infinity of I n sub k is less than 2 to the n.

Again, just because this is defined as an infimum, and so I can always for any little bit, if I go above it-- or a sub n, sorry-- if I go above it, then I should be able to find a collection of open intervals so that the sum of the links-- oh, God, what is that? Sum of links-- it's the end of the day. So I'm starting to make careless board mistakes.

And now, each of these collections for each n cover the a sub n's. And therefore, the union of the a sub n's is contained in the union of n in n, k in n of I in k. So this is a collection of open intervals covering this union is a countable, yes. Because it's in one to one correspondence with n cross n. n cross n is, again, a countable set.

So this is a countable collection of open intervals covering the union of a sub n. And therefore, the measure of a sub n should be less than or equal to the length of the sums.

And this is equal to sum over n sum over k, I in k. And this is less than sum n-- if I'm summing in k, remember I have that-- the sum in k is less than m star of a sub n plus epsilon over 2 to the n.

And now I'm summing in n. So n is going from 1 to infinity and so is k. And so this equals the sum of n, m star of a sub n plus epsilon.

So I started with-- ah, I keep making silly mistakes. Sorry about that. So I started with the measure of the union of the a sub n's. It's less than or equal to the sum of the lengths of these open intervals that cover this union. Because the outer measure is the infimum over all such sums.

So that's equal to this sum. How did we choose these open intervals? We chose them so that these lengths, the sum and k of these lengths gives me approximately the outer measure of each a sub n plus a little error. And I chose this error so that it's summable. If I would have just put epsilon here, I would have been summing something that can't be summed.

And in the end, I get this. So I've shown that for all epsilon positive, the outer measure of the union is less than or equal to the sum of the outer measures plus epsilon. I guess, less than. And now I just let epsilon go to 0. And I conclude that-- the bound that I want.

So this also gives a second proof, if you like, of what we proved a minute ago, that countable sets have measure 0. And if you-- and what we did right there, which is one element. We proved that the set containing one element has outer measure 0.

Now, a countable set is equal to a union of such sets. We've shown that the outer measure of this union, which would be this countable set, is less than or equal to the sum of the outer measure of the individual points in that set.

And we did, in this example, we did it for 0. But it doesn't matter that it was 0, that the outer measure of a singleton, a set with a single point, is 0. And therefore, the sum would be 0. And we would conclude that the outer measure of countable set is 0. So that's a second proof showing that the outer measure of a countable set is 0.

So we've shown that outer measure is-- so this is something that's defined for every subset of real numbers and that it satisfies three. Now, I'm going to leave-- it's actually going to be an exercise in the assignment, that outer measure also satisfies four. The outer measure of a set shifted is equal to the outer measure of the original set.

Why is this? A way to think about it is that if something holds for open intervals it should hold for outer measure. Because outer measure is defined in terms of sums of lengths of open intervals.

So if I take an open interval and shift it, then it's length does not change. The length of ab is the same as the length of a plus x to b plus x. So it's going to be an exercise in the assignment, that outer measure, in fact, satisfies four.

So what's left, and what we'll do next time, is show that if I is an interval, that the outer measure is the same as the length of the interval. And as intuitive as that is, it takes just a little bit of work to show-- not too much, but just a little.

And that will complete our construction of outer measure which is this set function which satisfies one through four but not three really. It satisfies almost three.

And then, once we restrict that outer measure to a certain class of subsets, certain well-behaved subsets, then we'll be able to get three. And that class of well-behaved subsets we will call Lebesgue measurable sets. All right. So I think we'll stop there.