[SQUEAKING]

[RUSTLING]

[CLICKING]

CASEY RODRIGUEZ: OK, let's continue our discussion of measurable sets. Let me just briefly recall for you, the end of last time, we discussed some general notions, some special collections of subsets of R, one of those being an algebra, which is closed under taking complements and taking finite unions. And then we said a collection of subsets of R's is a sigma algebra if it's also closed under taking countable unions.

And not every algebra is a sigma algebra. Why did we bring all this up? Well, we were in the middle of discussing Lebesgue measurable sets. So recall that we say E is Lebesgue measurable if for all subsets of R A, the outer measure of A is equal to the outer measure of A intersect E plus the outer measure of A intersect the complement.

So in some sense, a measurable set is one that divides sets nicely with respect to outer measure. And we denoted script M to be the set of all E such that E is Lebesgue measurable. And I'm going to stop saying Lebesgue measurable and just say measurable from now on.

And what we showed last time, we showed that M, the set of measurable subsets of R, do form an algebra. So it follows from the definition that if E is measurable, then E complement is measurable. But we also showed that if I take a finite union of measurable sets, or a finite collection of measurable sets, their union is measurable.

And what we're going to do today is we're going to show that the set, the collection of measurable, Lebesgue measurable, subsets of R form a sigma algebra. So they have this stronger property. And also that-- last time-- there's no way to write this in-- maybe not definition, but notation-- that B is the Borel sigma algebra, which is, recall, we proved it at the end of last time.

So we gave this as an example of a sigma algebra. This is the smallest sigma algebra containing all open sets. So any other sigma algebra that contains all open sets contains the sigma algebra B, the Borel sigma algebra.

So like I said, our goal for this lecture is we're going to show that this collection of Lebesgue measurable sets is a sigma algebra. We're going to show it contains the Borel sigma algebra. And then what is-- I mean, I can even state it now, or at least I'll just say in words.

What is Lebesgue measure? It's simply-- Lebesgue measure of a measurable set will be the outer measure of that set as long as it's Lebesgue measurable.

So what we're going to do is we're going to show that M is a sigma algebra.

Now, one preliminary remark that or lemma that we're going to prove is that so this condition 3 says that you need to prove that every countable collection is closed under taking unions, to ensure that an algebra is a sigma algebra. But in fact, you don't have to check it for an arbitrary collection. You really just need to check it only for a countable collection of disjoint subsets.

So that's the lemma that we'll use when we prove that the collection of Lebesgue measurable sets is a sigma algebra is the following-- so this is general. This has nothing to do, necessarily, with Lebesgue measure.

Let A be an algebra in-- so again, N here is a natural number-- a collection of elements of A. So each of these is a subset of R. Then there exists a countable collection F n of elements of A that are disjoint-- so if I take F n and F m, and n does not equal m, then their intersection is empty-- such that the union of the E n's equals the union of the F n's.

So what does this say? This says if we have-- let me make this a remark, then. What is the conclusion of this?

We only need to check this third condition for being a sigma algebra, that a countable collection is closed under taking unions, condition 3 for sigma algebra, for disjoint collections of elements. This follows immediately from this lemma because if I have any arbitrary collection of elements of the algebra, then their union is equal to a union of elements of A, where the union is now over elements that are disjoint from each other.

So this is the point. I only need to check the condition for an algebra to be a sigma algebra by checking that countable unions of disjoint sets remain in the algebra. So the proof is not very hard.

So let G-- let's call it something-- G n to be the union k equals 1 to n of E k. So then, these are growing because G n contains the first n E k's, and then n plus 1, I tack on another one. So G1 is contained in G2 is contained in G3.

And this is just kind of easy to check. And I'll leave it to you that the union of the E n's is equal to the union of the G n's. I mean, the G n's are really just unions of the E k's, finite unions of the E k's. So this is pretty clear.

I mean, if you actually want to sit down and do the argument, every element in G n is contained in E1 up to E n. And so every G n is contained in this union. And therefore, the union of the G n's is contained in this union. And then it's easy to go the other way around showing this union is contained in this union so that they're equal.

And now, I take F1 to be G1 and F n plus 1 to be F n plus 1 take away F n for n bigger than or equal to 1, or I should say-- sorry about that, that should be G n plus 1 take away G n. So what is this? I take the first set to be G1 and then F2 will be what's in G2 but not in G1.

So then, what is this? What do we get? We get that the union of any finite guy is equal to, in fact-- so this is pretty clear, again, because what am I doing here?

F2 will be what's in G2 take away G1. What's in F3 is whatever's in G3 take away G2. And their union is going to be the same as over here because I'm including G1 and so on. I hope this is clear.

And in fact, I don't know why I'm writing n. We in fact get this union is equal to this union. And again, I'm leaving some details out, but you can check that this union is contained in this union and then this union is contained in this union very easily. But I hope the idea is pretty clear because at each stage, you're taking whatever is in this set, cutting out what was appeared before it, in the sets before it.

So now, let's go back to measurable sets. We're now going to show that-- almost-- we are going to prove that the collection of Lebesgue measurable sets is a sigma algebra. But first, we need the following theorem, which is the following-- Let A be a subset of R. And this is essentially the whole game, as we'll see.

Let A be a subset of R E1 up to E n Lebesgue measurable, and not just measurable, but be disjoint measurable sets. So I only have finitely many of them. Then, the outer measure of A intersect the union of these guys is equal to the sum k equals 1 to n of the outer measure of A intersect E k.

So this shouldn't come as too much of a surprise because this is kind of close to saying-- I should say this is very true if say E1-- say you just have two sets-- E1 is E and E2 is E complement. Then that's just the definition of being measurable.

And you build up from that by using induction. That's the basic argument. Again, I'm sorry I keep moving away from the chalkboard because I'm used to lecturing to a classroom, even though it's been months now since I've been in a classroom with students. And because I'm a freak of nature who writes from my left hand, I have to write and then do this kind of move.

But anyways, we're going to prove this by induction. So the proof is by induction. So n equals 1. That's just one set. This is clear.

That's just the outer measure of A intersect E1 is equal to the outer measure of A intersect E1. So that's fine. So I'm going to put a check there without writing anything down.

So now, let's label this statement by star. So now, let's do the induction step. So suppose star holds for n equals m. And what we now want to show, meaning that if I have E1 up to up to E m disjoint measurable sets, then star holds.

Now, I want to show that statement holds for m plus 1. So let E1 up to E m plus 1 be measurable disjoint sets. So each of them-- they're pairwise disjoint.

Then, since E m plus 1 is measurable, what do we have? Well, first, before I use that, let me just note something. Since E m plus 1 is just disjoint from E1 up to E m, let A be a subset of R. We want to verify star for A, so I need to have an A.

So since E k intersect E m plus 1 equals the empty set for all k equals 1 to m, we get that A intersect the union k equals 1 m plus 1. What does this equal to? Or then I intersect that with E m plus 1, this is simply equal to A intersect E m plus 1 because E1 up to E m-- these are disjoint.

So when I transfer this through, I can write this intersection by bringing this intersect, each of these E k's, where k equals 1 to m plus 1. And then this is empty when k is not equal to m plus 1. So I just pick up E m plus 1 and I get that there.

But also, if I look at intersect complement, what is this? This is equal to-- now k, E k for k going from 1 to m, these are all contained in the complement of E m plus 1 because they're disjoint. So this is equal to just A intersect union A equals 1 to m E k.

Now, I got ahead of myself a minute ago, but now we're at this stage. Since E m plus 1 is measurable, the measure of A intersect this union A equals 1 m plus 1 E k, this is equal to the outer measure of A intersect k equals 1 m plus 1 E k intersect E m plus 1 plus the outer measure of A intersect union k equals 1 m plus 1 E k intersect E m plus 1 complement. Just using that E n plus 1 is measurable.

And now, we plug in what these things are. So this whole thing is equal to A intersect E m plus 1. So this is equal to the measure of A intersect E m plus 1. And now, this whole thing is equal to the measure of A intersect the union only going up to m.

And now, this is where we use the induction hypothesis because now we have a union of m disjoint measurable sets. So this is equal to plus sum from k equals 1 to m A intersect E k. So this is by our induction hypothesis. And combining these two terms is exactly what we want for n equals m plus 1, k equals 1 m plus 1 outer measure of A intersect E k. And that's the proof.

So now using this theorem and the lemma before it, we will prove that the collection m of Lebesgue measurable sets is a sigma algebra. We already know it's an algebra up to this point. But we just need to verify it's a sigma algebra.

So what's the proof? Now, again, based on this remark here, we only need to verify that if I have a countable collection of disjoint measurable sets, then the union is measurable. I don't have to check it for every collection of measurable sets, just disjoint measurable sets.

So we just need to check. So we've already checked it's an algebra. So by the lemma, the first thing we proved during this lecture, we just need to show m is closed under taking countable disjoint unions, meaning if I have a countable collection of disjoint measurable sets, I need to show the union is measurable. So let E sub n be a countable collection here-- n is a natural number-- of disjoint measurable sets.

So we need to verify the definition of being measurable. But remember, that reduces really to one inequality, since one of those inequalities is always clear. So let A be a subset of R, and let's denote E to be the union.

So what do we need to show? We want to show that the outer measure of A intersect E complement plus the outer measure of A intersect E-- E here, again, is a union-- is equal to the outer measure of A. But we always have the outer measure of A is less than or equal to this, so we just need to show this.

So let's do this. Let N be a natural number. So let N be a natural number by-- so since M is an algebra, this, a finite union, n equals 1 to capital N of E sub n is measurable. And therefore, if I want the outer measure of A, this is equal to the outer measure of A intersect n equals 1 to N E sub n plus the outer measure of A intersect n equals 1 to N E n complement.

Now, this finite union is contained in E. E is the total union. And therefore, the complement of this finite union contains the complement of E.

So this complement here contains the complement of E, and therefore, A intersect this is a bigger set than A intersect the complement of E. So this is bigger than or equal to the outer measure of A intersect union n equals 1 to N E n plus the outer measure of A intersect E complement-- again, because this finite union is contained in the whole union. So when I take complements, that switches around what's contained in what. So then I get A intersect E complement is contained in A intersect the complement of this finite union.

And that's good. So this is now up here. We wanted this to be smaller than the measure of A here. And now what do we have? We have the outer measure of A with a finite union of disjoint measurable sets.

And we can write this using the previous theorem as sum from n equals 1 to capital N outer measure of A intersect E n plus-- just keeping the second term. Now, this holds for every N. N was arbitrary.

So I can let capital N go to infinity. Remember, I've shown that the outer measure of A is bigger than or equal to this quantity here. So I can let N go to infinity to conclude that the outer measure of A is bigger than or equal to sum from n equals 1 to infinity of the outer measure of A intersect E n plus the outer measure of the A intersect the complement of the union.

Now, remember what we proved about outer measure-- that the sum of the outer measures is bigger than or equal to the outer measure of the union. So this thing is bigger than or equal to the outer measure of the union over all N A intersect E n plus A intersect E complement. And this is just equal to plus the outer measure of A intersect E complement.

So we've shown that the collection of all Lebesgue measurable subsets of R form a sigma algebra. So let me just-- maybe I should have said this at the start, but let me just pause here for a second. And maybe you're wondering why all this sigma algebra business anyways? Why have I imposed this condition? Am I just making this definition up at as I go along?

But it's kind of a condition that's forced upon us based on our expectations, in the following sense-- you remember one of the properties that we wanted of Lebesgue measure, or measure in general, was that the measure of a countable union of disjoint sets is equal to the sum of the measures. And I stated without proof-- but there is a proof in the textbook-- that you cannot have a measure defined on every subset satisfying those properties that we outlined.

So you have to have some collection of subsets on which you have these properties, that the measure of an interval is the length of the interval, and that's translation invariant, and that the measure of the union is the sum of the measures. That last statement, though, hidden in there, there's kind of a subtle assumption-- that if you have a measure defined on some collection of subsets, then that collection of subsets better be closed under taking countable unions. If I'm to be able to make the statement that I want, that the measure of a countable union of disjoint sets is equal to the sum of the measures, then hidden in there is that my measure has to be defined for a countable union of measurable sets.

In other words, if I have a countable collection of measurable sets, then for the statement I want to even make sense, I should have that the union of this countable collection of measurable sets is contained in the class of sets that I'm measuring. And so before we even discussed outer measure, all that, how you could maybe see coming that we would have some condition like the class of sets that we're going to measure will be a sigma algebra, or should be a sigma algebra, should be closed under taking countable unions. So maybe I was rambling, but hopefully, you got something out of that.

So we've shown that the set of measurable sets is a sigma algebra. Now what I'm going to show is that it contains the Borel sigma algebra. So remember, the Borel sigma algebra is the smallest sigma algebra that contains all open sets.

If I have any other sigma algebra containing open sets, that contains all open sets, then it must contain the Borel sigma algebra because the Borel sigma algebra is the smallest. And I should quantify smallest-- smallest meaning with respect to inclusion. If there's any other sigma algebra that contains all open sets, then the Borel sigma algebra B is contained in that sigma algebra.

But first, let's prove kind of a simpler case. So let me state this, and then I'll explain. For all a in R, the open interval a to infinity is measurable. So in the end, we want to be able to show-- so we already know M is a sigma algebra. If we can then show that every open set is measurable, then that means that M is a sigma algebra containing all open sets, and therefore, it must contain the Borel sigma algebra, which remember, is the smallest sigma algebra containing all open sets.

So to build up to showing that every open set is measurable, let's start with a very simple type of open set, half open or half infinite open interval. So let A be a subset of R. And let's write A1 to be A intersect a to infinity, and A2 be A intersect the complement, which is just minus infinity to a closed.

So what do we want to show? To show that this set is measurable, we want to show that the outer measure of A1 plus the outer measure of A2 is less than or equal to the outer measure of A because A1 is A intersect a set, A2 is A intersect a complement of that set. Now, if the outer measure of A is infinite, this holds regardless. So if the outer measure is infinite, done.

So suppose the outer measure is finite. So what we're going to do is we're going to show that this sum on the left-hand side is less than or equal to this plus epsilon, where epsilon is arbitrary. And then we can send epsilon to 0 to get the inequality. And everything will reduce to what we've done with intervals, as you'll now see.

So let I n be a collection of open intervals such that-- remember, the outer measure of A is the infimum of the sum of lengths of intervals covering A-- so such that sum over n length of I n is less than or equal to the outer measure of A plus epsilon. Define J n to be I n intersect A infinity K n to be I n intersect the complement of A infinity.

Now, the union of the J n's-- well, first off, let me say then each of these sets J n and K n are intervals. They're the intersection of two intervals, one open and the other closed, or empty.

Now, the union of the I n's covers A. So if I take the unions of the J n's, this will cover A intersect A infinity. Then A1 is contained in the union of the J n's. And similarly, A2 is contained in the union of the K n's.

And one more thing-- now, I n is an interval, and it's simple to check since just based on this being a finite interval that if I take the length of I n, this is equal to the length of J n plus the length of K n. So I take each I n, and I split it up into two subintervals. They're not going to be open intervals, necessarily.

This will be an open interval, this one not necessarily, but the sum of the lengths of these two intervals will add up to the length of the interval I n. That's clear. There's, if you'd like, J n and K n will be including that. And now, we're almost home free.

So A1 is contained in this union. A2 is contained in this union. So if I look at the measure of A1 plus the measure of A2, the measure of A1, since it's contained in this union, is going to be less than or equal to the sum of the outer measures of J n's. And the measure of A2, again, is contained in K, in the union of the K n's, so that's less than or equal to the sum of the measures of the K n's.

And now, bringing these two together in one sum over n, and using the fact that this is equal to-- so the outer measure of an interval is equal to its length, which we proved last time-- this is equal to sum length of the I n. And remember, how did we choose the I n? We chose this I n so that some of the lengths of the I n is less than or equal to the outer measure of A plus epsilon. So I should have added here, let epsilon be positive. Sorry about that.

So I've shown that for arbitrary epsilon positive, this number over here is less than or equal to-- I still haven't finished. Sorry, getting ahead of myself. That's what happens when you get excited.

And this is less than or equal to the outer measure of A plus epsilon. So we've shown that for arbitrary epsilon positive, this number is less than or equal to this number plus epsilon. So I can send epsilon to 0 to get that the outer measure of A1 plus the outer measure of A2 is less than or equal to the outer measure of A, which remember, that's what we wanted to show.

So we've shown that these open intervals from A to infinity are measurable. It's now not a long trip to saying that every open set is Lebesgue measurable. So theorem-- every open set is Lebesgue measurable, and thus, a Borel sigma algebra, which is the smallest sigma algebra containing all open sets, is contained in the sigma algebra of measurable sets.

So we've shown these types of intervals are open. Let's show finite open intervals are also-- I mean, we've shown these types of open intervals are measurable. Let's now show that finite open intervals are measurable.

So now, we have for all b in R that minus infinity to b, this is equal to the union n equals 1 to infinity of what? Of the union of these half closed intervals, which we can write as the complement of these types of open intervals.

Now, we just showed that these types of intervals are open, I mean, these types of open intervals are measurable. Therefore, their complement is measurable and the collection of measurable sets is a sigma algebra. So countable unions are also measurable. So we conclude that this guy is measurable.

And therefore, remember for an algebra, it's closed under-- for a sigma algebra, it's closed under taking complements and taking countable unions. But by De Morgan's laws, that means it's also closed under taking intersections. We conclude that for all a, b, and R, any finite open interval a, b, which is equal to minus infinity to b intersect a to infinity, this is measurable by what we just proved. This is measurable by the theorem we proved before.

And sigma algebras are also closed under taking countable unions. This is just a finite union. So this is also measurable.

Now, maybe you covered this in 100B, maybe you didn't. It's going to appear on the assignment. But I'll underline what's going to be on the assignment.

But since every open subset of R is a countable union of, in fact, disjoint open intervals-- open intervals meaning it could be a finite one, it could be one like that, it could be one like we wrote over there where it's a to infinity-- but every open subset can be written as a countable union of disjoint open intervals. And since open intervals, we've now concluded, are always measurable, that means their union is measurable. We conclude that every open set is measurable.

We now have a collection that we've said are measurable sets. They form a sigma algebra. They contain the Borel sigma algebra, which contains all open sets.

So now, we define Lebesgue measure. If E is a measurable set, the Lebesgue measure of E is denoted by m of E. And it's just given by the outer measure of E.

So like I said when we first started all this business, Lebesgue measure is simply outer measure restricted to a collection of well-behaved sets. And you see that here, is that Lebesgue measure is nothing but outer measure restricted to those sets which we call measurable. And measurable meant that they had this property that they kind of split sets evenly with respect to outer measure.

So immediately, we get to a few simple things. Theorem-- if A and B are measurable, and A is contained in B, then the measure of A is less than or equal to the measure of B. Why is this? Again, because outer measure satisfies that. m of A is just the outer measure of A, and that's less than or equal to the outer measure of B, which is, by definition, the measure of B.

So the point is that Lebesgue measure inherits many properties from outer measure. In particular, we have this. And then we also have, since we know-- let me make one more.

Every interval is measurable. Let's state it this way. If I is an interval, then I is measurable and the measure of I equals the length of I. So we've shown all open intervals are measurable. That's what we did a minute ago when we showed-- so first off, let me not get ahead of myself.

If I've shown that every I is measurable, then since Lebesgue measure is just a restriction of outer measure, and the outer measure of an interval is the length of the interval, this is immediate. So I just need to verify that every interval is measurable. Now, we've shown every open interval is measurable, but from there, it's not difficult to get that every interval is measurable.

So for example, if I take a closed and bounded interval, this is equal to-- what is this equal to? This is equal to b infinity complement intersect minus infinity a complement. So b infinity complement gives me b to infinity, including b. The complement of minus infinity to a is a to infinity, including a, and their intersection gives me a, b.

Now, open intervals are measurable. Therefore, the complement is measurable. So each of these things is measurable, and therefore, their intersection is measurable. So that's measurable.

And it's the same game if I take away one of these guys, except now, we would use kind of a trick like that. So let me do one of them. Let's say we look at a, b.

Well, I mean, I don't think we even have to do anything like that. Let's say I look at something like this-- this is equal to b intersect complement. So a half-open interval including a, not including b, is equal to minus infinity b intersect minus infinity a complement.

Because then, this gives me a to infinity, including a. And this is measurable, this is measurable, the complement's measurable, and therefore, the intersection's measurable. So this is measurable.

And those are the only two examples I'm going to do. And basically, by taking the complements of these, you get the other types of intervals. So this is a good one because this is one of the properties, remember, that we wanted of measure.

We at least wanted to be able to measure intervals. And we wanted the measure of an interval, or at least a Lebesgue measure of an interval, to be the length of that interval. And now, let's verify that other condition that we wanted, that the measure of a countable disjoint union is the sum of the measures.

So suppose E n is a countable collection of disjoint measurable sets. Then the Lebesgue measure of the union is equal to the sum of the Lebesgue measure of the sets. Now this, we have one of the inequalities here or we always have this is less than or equal to this simply because that follows from outer measure.

Remember, outer measure satisfied this with an inequality there, with M star and M star, but not equality. Specializing to these well-behaved sets gives us equality, as we'll see. So let me just reiterate that we do have one-- we'll prove this by showing one is less than or equal to the other and vice versa. We always have this is less than or equal to this because of outer measure.

So first off, we know that we have this countable union is measurable because M is a sigma algebra. We showed that already. And therefore, the measure of the union is, by definition, equal to the outer measure of the union, which is less than or equal to the sum of the outer measures of the E n's.

And remember, these are all measurable. So by definition, the outer measure is the measure. So this is what I meant by we always have one inequality or we already have one side of this inequality from outer measure.

So now we just need to show the opposite inequality. We now show sum n measure of E n is less than or equal to-- so how do we show this? Let m be a natural number. Then what is the measure of a finite union of these guys?

Well, I can write this also as the measure. So this is measurable, and therefore, I can write this as, in fact, R intersect. This is kind of a stupid way to write it, but I'm just doing it this way so that it looks like something we've already proven.

Earlier, we proved that for measurable sets, disjoint measurable sets, the outer measure of a set intersect a finite union of disjoint measurable sets is equal to the sum of these measures. And this is equal to E sub n. And the outer measure of a measurable set is, by definition again, the measure of the set. So this is equal to n equals 1 to E sub n.

So what we've shown is that in fact, for a finite disjoint union, the measure is equal to the sum of the measures. I mean, we already did prove that, except with an A there. But this gives us the opposite inequality once we realize that we have this sum which is equal to the measure of, and which is less than or equal to the measure of the big total union. Because this set is contained in this set.

So what I have is that N was arbitrary. I had that thing on the left-hand side is less than or equal to this thing on the right-hand side for arbitrary N. So now I let N go to infinity to conclude that-- as desired. So this is that other property that we wanted, that the measure of a disjoint union is the sum of the measures.

Now, there was that last property we wanted that measure is translation invariant. That will be in the assignment, so let me state it here. So what you'll prove in the assignment is if E is a measurable set and x is in R, then the shift of the set x E plus x, which is the set of all y plus x such that y is in E, is measurable. And measure of E equals measure of E plus x.

So this will be in the next assignment, which will be assignment, I think, 4, which is the third property that we desired. So Lebesgue measure, which is just outer measure restricted to the class of measurable subsets, which is a sigma algebra, satisfies the three major things we wanted of a measure. Unfortunately, the measure is not defined on all subsets, but it's defined on a large class of-- a very rich class of subsets of real numbers because it contains open sets, closed sets, and like I said, sigma algebras are closed under taking countable intersections and complements.

So you could take a collection of open sets and take its intersection, which is not necessarily an open set, but that would be in the sigma algebra. And then you could take a countable union of those types of sets and still remain in the sigma algebra. And then you could take complements of those types of sets and stay in the sigma algebra. So like my instructor said, if you can write down the set, chances are it's measurable.

So one last theorem we'll prove about measure, and then we'll call it a day, and call it for the theory of measure by itself. Then we'll move on to measurable functions, and then Lebesgue integration of measurable functions is the following, if you like continuity of measure, which is the following.

So suppose E k is a collection of measurable sets such that E1 is contained in E2 is contained in E3, and so on. Then the measure of k from 1 to infinity of E k-- this is equal to limit as n goes to infinity of the measure of k equals n of E k, which equals limit as n goes to infinity of the measure of E n. Now, I just need to show this is equal to this. And that's what I'll show, is that this is equal to this.

The fact that this is equal to this follows from the assumption that they're nested. E1 is contained in E2 is contained in up to E n, so the union is equal to E n. So I'm just going to show that these two things underlined in yellow are equal to each other. And then the fact that this is equal to that just follows from this assumption here.

So for the proof, I'm going to do this trick again where we're going to write the union, this countable union, as a union of disjoint sets. So we let F1 equal to be E1 F k plus 1 equal to E k plus 1 take away E k for k bigger than or equal to 1. And let me just remark, this is equal to E k plus 1 intersect E k complement, since they're nested, since E1 is contained in E2 is contained, and so on.

And note that since the E k's are measurable, this is measurable, this is measurable. Its complement is measurable. The intersection is measurable. So each of these are measurable.

Then, F k is a disjoint collection of measurable sets. And for all n in N, if I look at the union k equals 1 to n of F k-- so how am I building these guys? I take F1 to be E1.

F2 is going to be E2 take away whatever was already contained in E1. F3 is whatever E3 is take away everything that was already appearing in E2. So you can check that the union from k equals 1 to n of F k is equal to E n, and therefore also, that the union is equal to this union.

And now, we conclude that if I take the measure of k equals 1 to infinity of E k, this is equal to this union here, which is a union of disjoint measurable sets. So this is equal to the sum of k equals 1 to infinity of the measure of F sub k, which is equal to the limit as n goes to infinity k equals 1 to n measure of F k.

And again, if you like, I can rewrite this as limit as n goes to infinity, this finite sum of measures of these disjoint sets, as the measure of k equals 1 to n of F k which equals, as we noted right here, is equal to the measure of E n. So that takes care of the definition of Lebesgue measure. Next time, we will define Lebesgue measurable functions, which are, in a certain sense-- with respect to integration-- the analog of continuous functions.

So continuous functions have the property that if I have an open set in the target, so if F is a function going from x to y, if I have a open set in y, then the inverse image of that open set is an open set in x. Measurable functions will be similar to that, except now with measurable sets, but not quite. We're not going to ask that they take Lebesgue measurable sets to Lebesgue measurable sets, but Borel measurable sets to the inverse image should be a Lebesgue measurable set. We'll stop there.