[SQUEAKING] [RUSTLING] [CLICKING]

PROFESSOR: OK, so let's continue our discussion of normed spaces and Banach spaces. So last time, I introduced the space of bounded linear operators from one normed space from another. And we saw that when the target is a Banach space, then this space of bounded linear operators is also a Banach space.

So now, we'll look at another example of a way to obtain a normed space from a given normed space. So I'll be talking about subspaces and quotients, all right? So let me just recall briefly what a subspace is. You should remember this from linear algebra.

V will always be at least a vector space, usually a normed space. So we say a subset-- and I've been using this notation. It doesn't mean strict-- a strict subset. I'm just kind of used to using this notation from 18.100A. And the textbook use this notation for just being a subset, not necessarily a strict subset. So quick note about that. So a subset of V is a subspace if for all elements, two elements in the field of scalars and elements in W, their linear combination is also in W. So W is closed, undertaking linear combinations.

And so it's quite easy to prove-- and I'll leave this to you-- that the subspace of a normed space-- of a Banach space, sorry, is itself a Banach space, meaning it's complete with respect to the norm it inherits from V. It's a Banach space if and only if it's a closed subset of V with respect to this metric we have induced by the norm.

I mean, you can let me talk our way through it real quick. Assuming W is a Banach space, we want to show W is also a closed subset of V. So that means you need to show, for example, that-- one equivalent way of showing this is that every sequence that converges-- every sequence of elements from W that converges converges to an element of W.

Now, take a sequence in W that converges to some element in V. We now want to show that element's in W. Then that sequence converges in W-- or is Cauchy in W. And since W is a Banach space, there must be an element in W which it converges to. And therefore, by uniqueness of limits, that limit of that original sequence must lie in W.

And now, going the opposite direction, assuming the subspace is closed, we want to show that now this subspace is a Banach space, meaning it's complete. Well, let's take a Cauchy sequence in W. This is also a Cauchy sequence in V. Therefore, it has a limit. This limit has to be in W, since W is closed. And therefore, every Cauchy sequence in W has a limit in W. And therefore, W is a Banach space.

OK, so I didn't write out what the-- I didn't write out the details, but we talked our way through it. Since we're taking a trip down memory lane, I just recalled what a subspace is. Now, given a subspace, we can obtain another vector space with the data of the space and the subspace called the quotient.

So let me recall the quotient. So this is about briefly recalling what the quotient is. So let's take a subspace of V. We define an equivalence relation on V by saying that v is related to v-prime if and only if v minus v-prime is in W.

Now, what do I mean by equivalence relation? v is always related to v because the difference of v is equal to 0, which is an element of W because W is the subspace and therefore a vector space in its own right. If v is related to v-prime, then this implies v-prime is related to v, simply because we-- again, if this is in W, then it's minus is also in W. And so this is reflexivity.

And then we also have transitivity, that if v is related to v-prime and v is related-- and v-prime is related to v-double prime, then v is related to v-double prime. Just simply take v minus v-double prime and add and subtract v-prime. Then you obtain the sum of two elements in W which must be in W. So this is an equivalence relation.

And so then we define the equivalence class of v. This is the set of all v-prime in V such that v-prime is related to v. And then we define a new set, v-mod W, to be the set of all equivalence classes.

And instead of writing the equivalence class of v, we typically write v plus W instead of the equivalence class of v. So you think of the equivalence class of v as being v plus all elements of W. Or two elements are the same if they differ by an element of W.

So this is a set. But it becomes a vector space with the addition and scalar multiplication operations defined in a kind of natural way with v1. And scalar multiplication is just the equivalence class given by lambda times v. Now, you have to take a second and make sure that these operations are well-defined, which I'm sure you did in linear algebra.

What do I mean by well-defined? Well, these are two equivalent classes. So if I take two other representatives-- say v1-prime and v2-prime-- do I get the same equivalence class as I did with the unprime, meaning is v1-prime plus v2-prime plus W the same equivalence class as v1 plus v2 plus W? And this is not too hard to say-- or to see.

So v-mod W, which is typically how you pronounce this v slash W-- v-mod W is a vector space. And so note, we can identify W, so just the set W, with the equivalence class 0 plus W, which is also equal to little w plus capital W.

OK, so when we first started talking about norms, I also introduced what was called a seminorm. Now, a seminorm, recall, satisfied not all three properties a norm satisfied. It satisfies homogeneity, meaning how scholars pull out, and also the triangle inequality. But it didn't satisfy positive definiteness, meaning the seminorm could be 0 for certain non-zero vectors, OK?

How could that arise? How could a seminorm arise? Well, think of taking the maximum of the derivative of a function as being a norm or as being a potential norm. Then it satisfies homogeneity and the triangle inequality. But it is not a norm because the derivative of constant functions is 0. But constant functions aren't identically 0, if you think of them as elements of, say, the space of continuously differentiable functions.

But this next theorem says that if you mod out by the elements in a vector space on which the seminorm is 0, then you get essentially a normed space where the norm is given by the seminorm. So if you mod out by the elements with 0 seminorm, you get an actual norm space.

The statement is let-- let's take a seminorm on vector space V. Then we define E to be the set of all v in capital V such that this seminorm of these vectors is 0. This is a subspace of V. And if I define the following function on v-mod E-- so what is it? This is simply the norm of the representative v of this equivalence class.

So if I define a function like this, so then this function defines a norm on the space v-mod E. So in short, if I have a seminorm and I mod out by all the elements with seminorm equal to 0, then I get an actual normed space, where this norm is essentially the seminorm.

OK, so let's prove this. So why is this a subspace? Well, if I take two elements in this space of elements that have 0 seminorm and two scalars, and if I take the seminorm of the linear combination form, I want to show this is equal to 0. And this follows essentially by homogeneity and the triangle inequality.

This is less than or equal to-- if I apply the triangle inequality and then pull the scalars out, this is less than or equal to plus-- and if these both have seminorm equal to 0, this is equal to 0. And now, remember, a seminorm is always non-negative. So if I show something is less than or equal to 0, it's also-- then it must be equal to 0.

So first off, we define this function. So now, let's show that this defines a norm on v-mod E. We need to first show it's well-defined because I'm defining it in terms of a representative of this equivalence class, all right?

And what does this mean? i.e. If, v plus-- if I have two equivalence classes with-- or if I have the same equivalence class with two different representatives, v and v-prime, then-- and I should write E, not W. Then the norm of v equals the norm of v-prime. And therefore, this function is well-defined.

Now, how do we do that? Again, it's going to be essentially the triangle inequality. So suppose v plus e equals v-prime plus em meaning I have two representatives for the same equivalence class. And that means there exists a little e capital E such that v is equal to v-prime plus e.

Then if I take the norm of v, this is equal to norm of v-prime plus e. And by the triangle inequality, this is less than or equal to norm of v-prime plus seminorm of E. And now, since E is coming from this set of all vectors with seminorm equal to 0, this is equal to 0. And therefore, I get the seminorm of v-prime.

So I've shown that the seminorm of v is less than or equal to the seminorm of v-prime. Now, there was nothing-- this argument is symmetric in v and v-prime, all right? If v is equal to v-prime plus e for some element in capital E, so is v-prime. So I can also switch the-- I can also switch v and v-prime, meaning I've shown that v is less than or equal to v-prime and the norm of v-prime is less than or equal to v. And therefore, their two norms agree. So this function here is well-defined.

And so I'm going to leave it to you now to check that this function now, which is well-defined on v-mod E, is, in fact-- does, in fact, satisfy all the properties of a seminorm. I mean, it's non-negative. So from essentially the homogeneity and triangle inequality property of the seminorm, the triangle inequality and homogeneity of this function then follow. And we've essentially equated all elements with seminorm equal to 0 to 0. So this is why it's also positive definite. So I'm going to leave that to you, OK?

OK, now, there's one other way-- if we're given-- so in this process, we started with a seminorm on the space, identified the subspace of all, if you like, zero norm elements, and obtained a new normed space. But you could start off with a normed space, a closed subspace of that normed space, and obtain a new normed space on v-mod W, if w is a closed space, in similar fashion, where I define this norm this way. And well, the norm on that space won't be the same as this one. But that'll be in the exercises, OK? OK.

So that's about-- this is kind of concluding the elementary section of functional analysis, meaning the bare bones of Banach spaces and normed spaces. And now, with the rest of this lecture and then next lecture, we're going to get into the fundamental results of functional analysis related to Banach spaces. So these theorems that I'll now be stating and proving in the coming couple of lectures have names attached to them. So you should definitely know what they are, what they state, what they don't state.

But to prove them, I first need a result from metric spaces, which I did teach last semester. But we didn't cover this theorem. I can't remember if I covered it when I taught 18.100B. I don't think I did.

So this is about when can you write-- or in what way-- OK, so let me just state that them. And then I'll interpret it for you. So this is Baire's theorem. It also goes by the name of the Baire category theorem, although it has nothing to do with category theory as is, I guess, used today. I don't know if category theory gets used or if it just is created for itself. But Baire's category theorem is the only category theorem that I know.

So what does it state? If M-- so again, this is a theorem about metric spaces, so not necessarily normed spaces. So if M is a complete metric space and Cn is a collection of closed subsets of M such that M is their union-- so this is a-- so I have an N here.

So I should specify this is N as a natural number-- then at least one Cn contains an open ball B x, r. So recall this is a set of all y in M such that the distance from x to y is less than r. So at least one Cn contains an open ball could be stated more succinctly as at least one Cn has an interior point.

So sometimes in applications, these Cn's are not necessarily closed. But if M is equal to the union of these sets, then it's also equal to the union of their closures. And so what this theorem says is that if you can write a metric space as a union of sets, then the closure of one of those sets has to contain an open ball.

Now, let's think a little bit of what it means for-- or there's a specific terminology for when a closed-- when the closure of a set does not contain an open ball, that means it's nowhere dense. Or that's the phrase we usually use for that, is that it's nowhere dense. So Baire's theorem says that if you have M equal to the union of a collection of sets, then the closure of one of those sets has to be dense somewhere. Or you can't write M, a complete metric space, as the union of nowhere dense subsets.

And so this theorem is quite simple to state. It's quite very useful and, I mean, quite powerful in applications. In fact, you can use this theorem to give an alternative proof to something that hopefully you saw in your analysis class, which is that there exists a continuous function which is nowhere differentiable. Maybe I'll put that in the exercises. Maybe I'll just point you to it somewhere. We'll see.

All right, so the proof is by contradiction. I believe this is the first contradiction proof that we've done in the class. So in 18.100A and B, I mean, you start off-- everything is by contradiction. So let's suppose not, i.e. you can find there is a collection of closed subsets of M such that M is equal to this union. And what we're going to do is we're going to find a point that's not in this union and therefore not in M. And that will be our contradiction.

We'll use the completeness of M to obtain this point because how we're going to obtain this point is as a limit of a certain sequence. And to be able to say that this limit exists, we're going to show the sequence as Cauchy. All right, so we're going to build up this sequence inductively. And I will write this inductive proof kind of carefully this first time. And then in the future, I'm just going to say, OK, choose p1 this way, choose p2 this way. Continuing in this manner, we obtain a sequence of points, blah, blah, blah. But at this stage, I'll write this inductive proof kind of carefully.

OK, so suppose not. Since M certainly contains an open ball and C1 cannot contain an open ball, this implies M does not equal to the first closed set. Otherwise, the first closed set would contain an open ball. Thus, there exists an element p1 in M take away C1.

Now, C1 is closed. This implies that its complement is open. And therefore, I can find a small epsilon 1 such that the ball centered at p1 of radius epsilon 1 intersect C1 equals the empty set.

Now, I'm going to pick C2. Now the ball centered at p1 of radius epsilon 1 over 3, this is not contained in C2 because, again, this is by contradiction. So we're assuming that M is written as a union of closed sets and none of these closed sets contain an open ball. So this open ball cannot be contained in C2.

And therefore, there exists some point p2 in this open ball centered at p1 of radius epsilon 1 over 3 such that p2 is not in C2. Now, again, C2-closed implies that there exists an epsilon 2 less than epsilon over 3-- so we can make it very small if we wish-- such that the ball centered at p2 of radius epsilon 2 intersect C2 equals the empty set.

So now, I've picked p2-- p1 and p2. Now, at this point, I would usually say, continuing in this manner, we obtain a sequence of points. But let me write out the argument carefully.

Suppose there exists k points-- p1, pk and positive epsilon 1, epsilon k such that two things occur. So epsilon k is less than epsilon k minus 1 over 3 and-- which is less than epsilon k minus 2 over 3 and so on all the way down to epsilon 1 over 3 k minus 1. So if you like, k here is bigger than or equal to 2.

And pj satisfies-- pj is in the ball centered at pj minus 1 of radius epsilon j minus 1 over 3. And the ball centered at pj epsilon j intersects Cj equals the empty set. So let me star those two properties right there. We're going to obtain another point satisfying those two properties.

And I don't know why I'm being a stickler. It doesn't really matter. But let me put a 2 because that's not necessarily satisfied for p1.

But OK, now, I want to show there exists-- I can obtain a k plus first point satisfying those two properties. Again, this is just formally saying that once I've chosen p2, then I can choose a p3 doing this argument again. And it'll satisfy everything p2 satisfied with epsilon 1, except now with epsilon 2. But now, I'm just making it more formal.

So now, again, M is the union of all these closed sets, none of which contain an open ball. So since the ball of radius pk epsilon k over 3 is not contained in Ck plus 1, there exists an element pk plus 1 not in Ck plus 1. Or I should say-- over 3-- such that pk plus 1 is not in Ck plus 1.

So let me-- there's pk epsilon k over 3, Ck plus 1. So maybe they're not disjoint. Maybe there's some overlap. But then I can find a point pk plus 1 in there that's not in Ck plus 1.

So then there exists an epsilon k plus 1, which I can choose very small, say smaller than epsilon k over 3, such that-- again, because Ck plus 1 is closed, pk plus 1 is not in Ck plus 1. So I can find a small ball around pk plus 1 disjoint from Ck plus 1-- a ball-- we said Ck equals the empty set.

So then given k points, I can then choose a k plus first point satisfying these two bullet points here, all right? So by induction, we have found a sequence of points, pk, in M such that-- and epsilon k. So if you like, k starts at 2. Epsilon k and 0 epsilon 1 such that for all k, those two bullet points hold, such that-- which I denoted-- which I put a little star by, star [? hold. ?] So I'm talking about these two points. So when I say star, I mean these two statements here, OK?

OK, now, let's show that the sequence is Cauchy. So now, I claim-- how do we do that? This follows. So I'm not going to write out the full epsilon M argument. But I'm just going to write down the crucial estimate that proves this. This follows from the fact that for all k for all l, If I look at the distance between pk and pk plus l because in the end, when I look at-- if I want to show something's Cauchy, I need to take the difference of two-- or the distance between two points. One I can choose as the point occurring earlier in the sequence, and then one occurring later.

Anyways, then by the triangle inequality repeated, I can write that this is less than or equal to the distance from pk pk plus 1 plus the distance from pk plus 1 pk plus l. And now, I could do the triangle inequality again. And so I get that this is less than or equal to the distance from pk to pk plus 1 plus the distance from pk plus 1 to pk plus 2 and so on until I get to the distance between pk plus l minus 1 and pk plus l.

Now, by star, what's in yellow, pk plus 1 is in the epsilon k over 3 ball centered at pk. So this is less than epsilon k over 3. And the same thing here, less than epsilon k plus 1 over 3 plus epsilon k plus l minus 1 over 3. And now, all of those are less than-- if you go to the first one, it's less than epsilon 1 over 3 k minus 1. So then this is less than epsilon 1 over 3 k plus epsilon 1 over 3 k plus l, yeah.

And now, I can actually sum this up. This is strictly less than epsilon 1. And if I sum now from l equals k to infinity of 1 over 3 k, this is equal to 1 over epsilon 1 3 k 1 over 1 minus 1/3, which equals epsilon 1 over 2 3 to the minus k plus 1.

So the distance between a point pk and pk plus l is less than some constant times 3 to the minus k plus 1. So if k is very large, this is very small independent of l. So I shouldn't have used l here. And that's M, M. So this shows that the sequence is Cauchy.

So since M is complete, there exists a p in M such that these elements pk converge to p. And now, we're going to show this point p does not lie in any of the CJ's. And we'll do it by showing it's essentially in all of these balls, pj epsilon j.

All right, and it's kind a similar computation to what we just did. So now, for all k natural number, if you look at the distance between pk plus 1, pk plus 1 plus l-- again, so if I just go back to this computation right here, this, we proved, was less than epsilon k plus 1 times 1 over 3 k plus 1 plus-- or actually, 1 over 3 plus 1 over 3 squared plus 1 over 3 k plus 1-- k plus l.

Or I'm sorry, I think that's just l. And now, this thing is less than if I replace it by the infinite sum. So this is less than epsilon k plus 1 times the infinite sum again, m equals 0 to infinity 3 to the minus m, which equals what? So this is, again, equal to epsilon k plus 1 times 3/2.

OK, so we've proven that the distance between pk plus 1 pk plus 1 plus l is less than epsilon k plus 1 times 3 over 2. Let me take the limit as l goes to infinity. pk plus 1 plus l converges to the point p. So I get that the distance between pk plus 1 and the point p is less than or equal to 3/2 epsilon k plus 1. And remember, epsilon k plus 1 is less than a third epsilon k. So this is less than 1/2 epsilon k, all right?

And since pk plus 1-- remember, this is in the ball of radius 13 epsilon k. So I get that the distance between pk and p is less than or equal to the distance between pk and pk plus 1 plus 1 and p. We get this is less than or equal to 1/3 epsilon k plus 1/2 epsilon k. And this is less than epsilon k, which means that p is in the ball of radius pk of radius epsilon k, which by the second property up in star-- the fact that all these balls are disjoint from-- each of the balls is disjoint from Ck means p is not in Ck. Now, k was arbitrary here. So we conclude that p is not in this union, which is equal to M. And that's a contradiction.

So I mean, I said the strategy at the top of the proof, the technical argument maybe muddled what was going on. But again, the point is you can build-- if this conclusion does not hold, you can build a sequence which has this disjointness property from this collection of open sets such that the sequence is Cauchy. And because M is a complete metric space, you can then extract a limit p. This limit p will have all the properties that these pk's had, namely that they're not in any of the Ck's. And therefore, this point p does not lie in the metric space, which is nonsense. So we arrived at our contradiction. And therefore, Baire's category theorem is proven.

OK, so that's Baire categories theorem. Let's now use this to prove some fundamental results of functional analysis. So the first result we'll prove is what's called the uniform boundedness theorem, which says that if you have a sequence of linear-- bounded linear operators on a Banach space, then pointwise boundedness implies uniform mindedness or uniform boundedness in the operator norm.

So let B be a Banach space. And let Tn be a sequence of bounded linear operators, which we denoted last time by this script B, to some other normed space, V. So I didn't write that, but V is a normed space.

If for all b in capital B, sup n of Tn v-norm is less than infinity-- so if I assume that this sequence is pointwise bounded-- pointwise bounded, then I can conclude that they're uniformly bounded, namely the sup of the operator norms are bounded.

So we're going to use the Baire category theorem. So we're going to write the Banach space B as a union of closed subsets. What will be these closed subsets? Remember, we're trying to find a uniform bound on the Tn. So just playing around, so let's define the subset of B to Ck. This is a set of all elements b in B such that the norm of B is less than or equal to 1 and sup n Tn of B is less than or equal to k.

All right, so first off, these are closed. So I should say, what is k? k here is a natural number. Ck is closed. Why is this? Well, we need to show that a sequence converging-- a sequence of elements from Ck that converges converges to an element of Ck. So if Bn-- this is a sequence of elements in Ck-- and Bn converges to B, then the norm of b is equal to the limit as n goes to infinity of the norm of bn. And each of these is less than or equal to 1. So the limit has to be less than or equal to 1.

And so I shouldn't-- maybe I'll use AI a-- I have the Tn's. I have the Bn's. So let me use M. And for all M in natural number, Tm applied to be and normed, this is equal to the limit as m goes to infinity of Tm Bn in norm because these are bounded linear operators and therefore continuous.

OK, so Tm of B is equal to the limit as n goes to infinity of Tm of Bn. And now, the norm of Tm of Bn is always-- well, this is always less than or equal to k because the Bn's are in Ck and sup of all these guys is less than or equal to k. So that's also less than or equal to k. Thus, B is in Ck, which implies Ck is closed.

Now, the closed ball-- or I shouldn't say-- it's not the closure of the open ball, but the closed ball, b in capital B, such that b is less than or equal to 1. This is equal to the union over all k of Ck. Why?

So first off, each of these is contained in this closed ball. But I'm assuming by star that no matter what b is in the Banach space, sup over n of these is less than infinity. So there's always some integer k given a b-- so given a certain b in this set, I can always find a large enough integer such that this is less than that integer. So it has to-- every element in here has to lie in one of these sets C sub k So this is equal to the union.

So this is a complete metric space because it's a closed subset of M. So this is a complete metric space written as the union of a-- a union of closed subsets.

So by Baire's theorem, there exists one of these sets containing an open ball of the form B, B0, delta 0. So one of these Ck's contains an open ball. Now, we're going to show-- we're going to use this k to derive a uniform bound.

So if b is in B and-- let's see. So let's write it this way. If b is in the open ball of radius delta 0 centered at 0, so this means i.e. the norm of b is less than delta 0. Then b0 plus b is in this open ball centered at B0. And therefore, sup n of Tn of b0 plus b norm is less than or equal to k. So this ball is contained in Ck. And therefore, if I take the sup over n of T sub n applied to this thing, this is less than or equal to k.

But then I conclude that sup over n of Tn applied to B, which is less than or equal to sup n-- so I'm going to add and subtract B0. So let me actually write this as sup of Tn B0 plus Tn B0 plus B. This is less than or equal to by the triangle inequality. And then carrying the sup through, this is less than or equal to sup n Tn B0 plus sup n Tn B0 plus B.

And so B0 is certainly in this ball, which is contained in Ck. So this is less than or equal to k. And this is still in that ball of radius delta 0 centered at B0. So it lies in CK. And therefore, this sup is also less than or equal to k equals 2k.

So I've shown that if I take any element in the open ball of radius delta 0, then the sup over n of Tn of B is less than or equal to 2k, all right? And now, it's just a simple rescaling argument to show that the sup of the norms are now bounded.

So suppose norm of B equals 1. Then the norm of Tn of delta 0 over 2 applied to B is less than or equal to-- so for all n certainly less than or equal to the sup over n of all of these, which is less than or equal to 2k because this is an element with norm delta 0 over 2, which is less than delta 0. So this is less than or equal to 2k, which implies-- so, again Tn is linear. So I should have reversed this.

Let M be a natural number. Let B equal to 1. Yeah, OK. So then Tn applied to B is less than or equal to 4k over delta 0. Now, this holds for all B with length equal to 1. And therefore, the operator norm, which is the sup over all B, would normally equal to 1, is less than or equal to 4k over delta 0. This holds for all n. And therefore, the sup over n of the operator norm is less than or equal to 4k over delta 0, and therefore giving us the uniform bound.

I have time for this? OK, I don't think I have time for the entire proof of what's to come. So I think we'll just stop there for now.