[SQUEAKING]

[RUSTLING]

[CLICKING]

PROFESSOR: OK, so we're going to complete our discussion of Lebesgue measure and integration by discussing the big Lp spaces, which was kind of the whole point of this endeavor, which was to find, in some sense, the complete space of integratable functions, whatever integratable means--

I mean, we had to define an integral-- containing the space of continuous functions with norm given by, let's say, the integral of the p-th power of the continuous function.

OK, so at the end of-- so at last lecture, we introduced the general Lebesgue integratable-- I mean the class of Lebesgue-integratable functions and the Lebesgue integral.

And we proved that-- we proved the dominated convergence theorem, and one consequence of that was the fact that if I have a continuous function on a closed and bounded interval a, b, then the Lebesgue integral of that continuous function equals the Riemann integral of that continuous function.

So you know how to compute integrals-- or the Lebesgue integral for every function that how to compute the Riemann integral for, which are mostly continuous functions. Now this can be strengthened. One can show that, in fact-- so we're not going to cover it in this class.

You'll see it in another class that is devoted maybe just solely to measure theory for a longer bit, of time but one can show using the dominated convergence theorem that, in fact, every Riemann-integratable function-- not just continuous, but every Riemann-integratable function on a closed and bounded interval is Lebesgue-integratable, and that the Riemann integral equals the Lebesgue integral, even just for more general Riemann-integratable functions.

And you can also, now using the machinery that we built up-- I don't know, maybe I'll put this in the assignment maybe not. That you can completely characterize those functions which are Riemann-integratable. And the statement is that a function-- a measurable function, say, is Riemann-integratable if and only if it is continuous almost everywhere.

I'm not saying that it's equal to a continuous function almost everywhere. I'm saying that it's continuous at almost every point in the interval.

OK. Now let's move on to analogs of the little Lp spaces that we saw earlier in the lectures and on the assignments. So these are usually referred to as the big Lp spaces. And so to define these, let me first define what will be, in the end, a norm.

So if f from a measurable subset of R to the complex numbers is measurable, and 1 is less than or equal to p is less than infinity, then we define the following possibly-- or the following extended non-negative real number, norm Lp of E. This is defined to be the integral over E of f raised to the absolute value raised to the p to the 1 over p.

Now this is meaningful because no matter what f-- how f has behaved, because the absolute value of f raised to the p, this is a non-negative measurable function. So we can always define what the Lebesgue integral is. So this may be either infinite or finite, but it's a non-negative number, extended real number.

And so maybe you ask, what's going on? Why did I leave out p equals infinity? We have a different definition for equals infinity, just like we had a different definition for the little l infinity. We define this quantity here, which I'm going to go ahead and start referring to as the Lp and L infinity norms even though I haven't proved their-- a norm yet on what space either.

This is defined to be the infimum over M positive such that the measure of the set x and E such that f of x is bigger than m equals 0.

So what does it mean for m to be in this set? This means that f of x is less than or equal to m almost everywhere. And then I take the minimum of all such-- almost everywhere upper bounds. And this is usually referred to as-- what is called the essential supremum of f of x.

So just a little mini theorem about this L infinity norm here, what you'll see-- well, I guess you'll be seeing these lectures after the first exam. So you saw this guy actually on the exam, and you proved one of these facts. The other I will put on a future assignment.

If f from E to C is measurable, then the absolute value of f of x is less than or equal to gL infinity almost everywhere on E.

And another fact, if f-- if E is equal to a, b and f it is continuous on a, b, then this essential supremum is equal to, in fact, just the usual what we call the L infinity norm which, remember, was the sup over x and a, b absolute value of f of x.

So why do I state this? Because the L infinity norm bounds f from above for almost every x in the same way that the little l infinity norm for sequences bounded, the sequence-- every entry in the sequence-- for every entry in the sequence. But now for the essential supremum, we have just an almost everywhere statement.

But this norm is the same as the L infinity norm or the infinity norm for continuous functions. So it shouldn't be something that's too crazy. OK.

So now I'm just going to state a couple of theorems because you already gave the proof-- the proofs, I should say, when you did-- I think it was probably the first assignment, when you did the corresponding statements for little lp spaces, except now you replace the-- for those, you replace an integral with a sum. I mean, you should always think of an integral as a sum.

So we have the following two theorems, two inequalities. We have Holder's inequality. If one is between p-- if p is between 1 and infinity and q is the dual exponent to p, meaning 1 over p plus 1 over q equals 1, and you have f as an Lp of E and g is an Lp of E, then f times g integrated over E-- well I haven't even said-- sorry. Getting way ahead of myself.

So-- and if f and g are two measurable functions, then the integral over E of f times g absolute value, this is less than or equal to the Lp norm of f times the Lp norm of g.

Now of course, this inequality is only interesting if the right hand-side is finite. If this is infinite, then this is all vacuously true.

So this is the analog of the Holder's inequality which you proved for sequences where we had a sum here instead of the integral and where we had a sum here instead of the integral. And it's proven essentially the same way. You just replace a sigma with a little s.

And so from Holder's inequality, you obtained Minkowski's inequality. If p is between 1 and infinity, and f, g are two measurable functions, then by-- take the Lp norm of f plus g, this is less than or equal to the Lp norm of f plus the Lp norm of E. And again, you proved this exactly the same way as you did for the little Lp spaces using Holder's inequality.

Of course, that requires a slightly different argument for p equals infinity you know for this essential supremum, but in fact, that's what you did in the exam you took a few days ago.

So I've been calling these things a norm even though I haven't proved their norm yet and on what space are they a norm, so now I'm going to do that.

So when it's clear-- also let me make a small remark. I'll denote this thing just by shorthand with just a p. And it should be clear from the context what set I'm taking this norm over. Or what set am I taking this integral that defines this norm over.

OK, so now let me define the actual space that this will be a norm on. And it involves a slight abuse of terminology and notation in the end, which is just tradition, not just in this subject, but-- I mean, abuse of notation is tradition in all of math.

So for 1-- for p between 1 and infinity, we define the space Lp of E-- so E here is know-- if I don't say it, it's always a measurable subset of R. This is the set of all functions from E to C, which are measurable, that have finite Lp norm.

Now let me make a second caveat to this space. So as I've written it down now, it's a space of functions. And I'm going to keep referring to it as a space of functions. I'm going to keep referring to elements of it as functions. But the actual space itself is not a space of functions, it's a space of equivalence classes in order for this quantity, which I keep calling a norm, to actually be a norm on this space.

So let me add here-- if we consider two elements of this set, the-- I shouldn't say equal, but to be the same element-- and let me give two elements, say f and g in Lp, to be the same element if f equals g almost everywhere.

So as I've written it down, Lp of E consists of all measurable functions with finite Lp norm, and now I'm saying that I will consider two elements in this space to be the same-- to be the same element if they equal each other almost everywhere.

So strictly speaking, let me just make this as a remark. This means an element of Lp of E is an equivalence class of the form-- so little brackets f to indicate the equivalence class. I haven't told you what the equivalence relation is, so I'll explain that as I describe the equivalence class.

So the equivalence class of f is equal to the set of all function g, which are measurable. So Lp of E is really a set of all equivalents-- is a set of equivalence classes of measurable functions with finite Lp norm where two equivalence classes are equal if and only if the representative from the first equivalent class equals the representative of the second equivalent class almost everywhere.

So this is what Lp of E is. Now why am I making this-- why am I adding this caveat that you have to consider two elements to be equal almost everywhere? Because this is what allows me to put-- or say this Lp norm is an actual norm. Otherwise it is just a semi-norm if I really consider Lp of E to be this space of actual functions.

So again, this is a small point, that Lp of E is, in fact, a set of equivalence classes where two functions are two equivalence classes or equal if and only if the representative functions equal-- are equal almost everywhere.

But now that I've explained all of that and carefully told you what Lp of E is, it is customary to never refer to them as equivalence classes or elements of Lp of E as an equivalence class. We usually still just refer to them as functions. Let me make that point clear.

So rather than speaking of an equivalence class or elements of Lp of E as an equivalence class, I will just refer to them as functions with the understanding that two functions in Lp of E are considered to be the same function, the same element if they equal each other almost everywhere.

Now maybe you think this is a little weird, but you've been doing this your whole life already, because the rational numbers themselves are defined, if you look back into algebra or if you're taking algebra now, when you actually sit down and construct the rational numbers, they're constructed as equivalence classes of pairs of integers.

But you don't think of them as that, you think of them as 3 over 2. Not the equivalence class of 3, 2. So again, two elements in Lp of E. I mean, we think of Lp of E as a set of functions with finite Lp norm, and with the caveat that two functions-- or two elements of Lp and E-- Lp of E are equal-- are the same element if the two functions agree almost everywhere.

OK, now with that minor detail out of the way, let's move on to-- let me state the theorem that-- so Lp with the natural scalar multiplication where you just multiply-- the scalar multiple of an element as just multiplying the function by a scalar.

And the sum of two elements is just the pointwise why some of the two functions. So with the obvious definitions of scalar multiplication and addition. Operations is a vector space. Moreover, this function which I've been referring to as a norm is actually a norm on Lp.

Now I'm not going to-- I'm only going to prove part of this because to verify something as a vector space, you have to verify the operations are-- satisfy certain properties. And when I state the-- or give the proof of what I'm going to prove out of this theorem, this will be the last time I actually refer to the fact that these are equivalence glasses and not functions, but just to make a point.

Again, this space, strictly speaking, is a set of equivalence classes. So one would have to check that addition and scalar multiplication are well-defined on this set, meaning if I have two representatives of the same equivalence class and I multiply one by a scalar and the other by a scalar, then I get the same equivalence class in the end, which is easy to check, and the same for addition.

So again, in a strictly speaking sense, that Lp is a set of equivalence classes, these things are well-defined, scalar multiplication and addition. So let's check that Lp-- or that this quantity here is a norm. And I'm just going to--

So first off, note that taking the Lp norm of an element of Lp is actually well-defined. Remember, so this is going to be the only theorem and proof where I actually refer to the fact that Lp is actually a set of equivalence classes, and then after that we just won't do that anymore and we'll just think of Lp as a set of functions where two functions are the same element if they equal each other almost everywhere.

But first thing to note is that this is actually well-defined on this set of equivalence classes. So note, if I have two representatives of an equivalence class, then by what we developed for the Lebesgue integral--

So f, take the absolute value raised to the p is also going to be equal to g absolute value rates of the p almost everywhere, and therefore these two integrals are going to equal each other. These integrals are real numbers. They're not equivalence classes of real numbers, this is a real number, a non-negative real number.

And therefore, if in the notation I had before, if I have an equivalence class and I have two different representatives of this and I define the Lp norm of this equivalence class to be the integral of the representative, I get the same number regardless of the representative for that equivalence class.

I.e.-- OK? So this function is defined by-- if I take that Lp norm of an equivalence class given by just the integral of the representative, this is well-defined. If I take two different representatives, I get the same number out.

Now this quantity here equals 0 if and only if-- by what we've developed in our theory of integration, f equals 0 almost everywhere. So the Lp norm or the-- so I should say this equals 0 almost everywhere-- i.e., f equals 0 almost everywhere. But possibly not everywhere. The equivalence class of this function is equal to the 0 element.

So this was the whole reason why I went through this trouble of actually explaining to you why, strictly speaking, the most rigorous way of defining Lp is as a space of equivalence classes, even though we won't think about them that way or talk about them that way in the future, that's, in fact, what they are.

Because then, this quantity here, this Lp norm is, in fact, a norm. If the Lp norm is 0, then that element is the 0 element. OK. So that's about the last time I'm ever going to refer to the elements of Lp as equivalence classes. I will now be referring to them as functions.

But homogeneity and the triangle inequality for the Lp norm, then follow, simply from the definition, for homogeneity, the fact that a scalar-- that the Lp norm of a scalar multiple of f is equal to the absolute value of the scalar times the Lp norm of f, and the triangle inequality follow from the definition and Minkowski's inequality.

OK. So Lp is the space of measurable functions with finite Lp norms. See? I'm Already not going to refer to it ever again as the space of equivalence classes. You should think of them as space functions, just with two elements equal if these functions equal each other almost everywhere pointwise on E.

Now we come to the big question. We now have this norm space Lp of E corresponding to all these functions that have finite Lp norm. First off, is this non-empty? Let's maybe give you the simplest example. In fact, let me prove the following simple theorem.

So-- which is the following. Let E be measurable, then f is an Lp of E if and only if the limit as R goes to infinity over-- of the integral of minus R to R intersect E of f raised to the p is finite.

Notice, as R going to plus infinity-- so maybe let's not make it R. We can make it n we're in as a natural number. Then this is an increasing sequence of numbers. So let's give the proof real quick.

Let's assume f is in Lp implies that this quantity here is finite. Since the sequence of integrals over minus n to n intersect E f raised to the p, this is an increasing sequence. Because at each step, at each entry, I'm taking the integral of this non-negative quantity over a bigger set.

So this is an increasing sequence. So, in fact, this limit always exists, so it was meaningful to actually refer to this limit. Since this is an increasing sequence, limit as n goes to infinity minus n intersect E actually exists as a possibly extended real number.

Now since for all n, we have that the integral of minus n to n intersect E raised to the p is less than or equal to-- well, in fact, I'm being a little inefficient here. Let's just erase one bit here. OK. Yeah, let's do it this way. We'll do it much faster.

Now note that intersect E f raised to the p, this is equal to integral over E chi minus n to n f raised to the p. So I can think of this as-- I can think of what's here in the integrand as a function for each n.

So since this is a sequence that is pointwise increasing, and for all x in E, we have limit as n goes to infinity of chi minus n to n, f of x raised to the p equals f of x raised of the p.

This-- by the monotone convergence theorem, the integral of the limit as n goes to infinity, which is just f raised to the p, is equal to the limit as n goes to infinity of the integral of this quantity here over E, which is, again, the interval over minus n to n intersect E raised to the p.

And therefore, this quantity here is finite if and only if this quantity here is finite. And therefore, f is an Lp if and only if this quantity here is finite, and in fact, they equal each other.

So using that theorem, if you like, you can prove-- and I'll leave it to you-- that if f from, let's say, R to C is measurable and there exists a constant non negative and q bigger than 1 such that for almost every x in R, f of x is less than or equal to a constant times 1 plus x raised to the minus q, then f is an Lp of R for all p bigger than or equal to 1.

So how do you do that? So OK, maybe I'll just indicate why we use the previous theorem and look at the integral from minus n to n intersect R. So this here, by this estimate, is less than or equal to the integral from minus n to r times constant 1 plus x to the minus q.

Now this is a continuous function over a compact-- or closed and bounded interval, so this is equal to its Riemann integral. I will often, on exams and so on, right the Lebesgue integral also in this form, though.

So I don't want to say that I'm just-- you're going to use this kind of notation for the Riemann integral. But anyways. And I leave it to you to show that as long as q is bigger than 1, this is less than or equal to-- so I should put p times q. p times q.

That as long as q is bigger than 1, this integral here is less than or equal to some constant depending on p.

OK. So there's many functions that are in Lp, so it's not exactly a trivial space. But what kind of space is it? Now let me state the following. That, in fact, this is what you proved in the assignment right before -- or the assignment I at least assigned before the exam, which was following.

Let a less than b-- less than b. And p between 1 and infinity. So in fact, you did it for L1, but the same proof carries over for Lp. fE in Lp of a, b. So I keep adding stuff and epsilon would be positive.

Then there exists a continuous function in a, b, which I can also impose vanishes at the end points and is close to f and Lp norm. So what this states is that the space of continuous functions is dense an Lp of a, b.

And it's a proper subset of Lp of a, b. I can find elements in Lp that are not continuous or even not equal to a continuous function almost everywhere. So it is dense and also proper.

And now the final theorem will prove about integration in Lp spaces is that Lp is complete. So this is due to Reece and-- let's see. Does this have a C or is it just-- yeah, it does have a C. This is a Banach space. For p between 1 and infinity. Including 1 and infinity.

Now I'll give the proof for p between strictly less than infinity. So p equals infinity. This will appear in an assignment. So we'll do the case of p between 1 and infinity.

So how are we going to do this? We are going to, in fact, use that criterion from several weeks ago about when is a norm space a Banach space? So we proved this equivalent criterion-- so remember, a Banach space is a norm space that's complete with respect to the norm. So you would have to check that all Cauchy sequences in the space converge to something in the space.

But now we came up with this other criterion-- I shouldn't say we did. Somebody did, and then I showed it to you, that an equivalent way to prove that is to prove that all absolutely summable series in the norm space are summable.

Remember, absolutely summable means that some of the norms is finite. So that's what we're going to use-- or that's what we're going to do. We will show that every absolutely summable series is summable.

So, suppose I have a sequence in Lk, it's a sequence in-- f is an Lp, not Lk. That form an absolutely seminal series. So such that sum over k of-- if I take the LP norm of fk. So this is now a series of non-negative numbers, let's assume this converges, meaning this I'm writing it as this is finite.

So, in fact, this equals a-- let me call this a convergence series something. Call it m, which is a finite number. All right, so we have this absolutely summable series, and we want to prove that now the series, sum of fk's, converge to something in Lp.

And what do we want to show now? So let me just-- so that that's clear ahead of time, we want to show there exists a function in Lp such that k equals 1 to n. The partial sums converts to f as n goes to infinity and Lp of eie.

Equivalent way of writing that is that the limit as n goes to infinity of the norm, some k equals 1 to n of fk minus fp equals 0. That will show that every summable series is-- absolutely summable series is summable.

So we have to identify a candidate f and then show that the norm of-- that this norm here goes to 0 as n goes to infinity. OK, so define gn from E to-- so it's a non-negative number-- or it's a non-negative function-- by gn of x equals sum from k equals 1 to n of the absolute value of fk of x.

This is, again, a measurable function because it's the sum of measurable functions. So what do we know? By the triangle inequality, we know that if I take the Lp norm of gn, which is this finite sum-- so by the triangle inequality for the Lp norm, this is less than or equal to sum from k equals 1 to n of the Lp norms of the fk's.

And this is a partial sum corresponding to the series of the norms of fk's which sum to M. So this is always less than or equal to M, which we're assuming, again, is a finite number.

And therefore, which implies by Fatou's lemma that the liminf as n goes to infinity of g sub n over E, which is equal to--

Now for each x, this-- so as n goes to infinity-- for each x as n goes to infinity, this converges to something. It's either finite or equal to infinity, so it always converges. So this is equal to the infinite sum. I should say, let's raise this to the p. Which is equal to-- so-- OK, I kind of got this backwards. I should have said this is equal to this.

But anyways, by Fatou's lemma-- so let me reverse these things. So by Fatou's lemma, so I have this is equal to-- so this does not used Fatou. It just uses the definition of gn. This is less than or equal to-- now I'm using Fatou's lemma. Integral over E gn raised to the p.

And now I use that bound which I have right here-- this is always less than or equal to-- remember, the Lp norm of g sub n was less than or equal to M. So raising that to the p power, I have this is less than or equal to M to the p.

So I started off with the integral of this non-negative measurable function, which is the series of f sub k raised to the p, and showed that that integral is finite.

Thus-- so by another theorem that we proved from integration, if I have a non-negative measurable function that has finite integral, then that measurable function has to be finite almost everywhere. So thus, this quantity has to converge or is finite for almost every x in E.

So, what I get is that for almost-- let me include an x in here. So I've proven for almost every x in E, fk of x is absolutely convergent. So this series is absolutely convergent. And therefore, it converges for almost every x in E.

And I'll define my function away, right? Because in the end, remember, we're trying to find a function f so that These partial sums converge to f and Lp.

So if we can define f at somehow as the almost everywhere pointwise limit of s sub k, maybe we can use the dominated convergence theorem in some way, and that'll be what we do. So define f of x to be exactly what I get from this convergent series when it converges, absolutely.

So if-- 0 or otherwise. And I'm going to define g of x to be, again, if you like, it's when the g sub n's converge. So if this is finite, and 0 otherwise. So now I have these two measurable real value functions. And this guy will end up being what the fk's converged to.

So then we have a couple of things. Then limit as n goes to infinity of sum from k equals 1 to n, fk of x-- you can put a minus f of x-- equals 0 almost everywhere on E. Why is that? This is just because f of x is simply defined to be the infinite sum when the sum is absolutely convergent, which is almost everywhere by what we've done.

What else? If I-- so this is one crucial thing. And minus f of x, this is less than or equal to g of x-- so it's the p raised the p almost everywhere on E. OK, why is this?

Holds, for example, when the series is absolutely convergent. Because then f of x is equal to the infinite sum of the fk's of x. And so by the triangle inequality, the absolute value of that difference is going to be less than or equal to the sum of the absolute values. That's by definition equal to g sub n or g sub n plus 1. g is the limit of the g sub n, so the g sub n's are increasing. So that always sits below g of x.

Now, by-- I should say, by this, I don't want to pull it up. So let's just say, since we proved that the Lp norm of this is less than or equal to M, which this here is equal to g almost everywhere, this implies that the LP norm of g is equal to this Lp norm, and therefore is less than or equal to M And therefore, that means the Lp norm raised the PE is finite.

Moreover, what else do we deduce? We also deduce from this that f, which is bounded above by almost everywhere by this quantity, this is less than or equal to-- which is equal to g almost everywhere, the Lp form of f is less than or equal to the Lp norm of g, which, as we said, is less than or equal to M-- i.e., f is an Lp of E.

So again, almost everywhere we have that f which is equal to the sum without the absolute values. That's always less than or equal to an absolute value. The sum from k equals-- the sum over k with absolute values inside. And we know that the Lp form of this is finite, we proved that earlier, which tells me that the Lp norm of g is finite, because g is equal to this quantity almost everywhere. It's 0 on a set of measure 0, which we just threw in there to make it finite.

So we have f is an Lp, we have g is an Lp. So at least this can be a possible candidate. And we have these two facts here.

Now we apply the dominated convergence theorem using this fact, this-- we have converging to 0 almost everywhere on E. And since that's true, I can put, if you like, raised to the p. So this quantity here is converging to 0 almost everywhere. This quantity is also bounded above by this quantity here on the right, which is integrable. It has the integral of g to the p is finite.

So by the dominated convergence theorem, I can conclude that the limit as n goes to infinity of sum from k equals 1 to n of the fk's minus f raised to the p over E, this equals the integral of the limit, which, remember, is 0. I.e., we've shown that. Which is what we wanted to prove.

Now of course, you need a different argument for p equals infinity, and it's a little bit simpler. So we've proved that Lp is a Banach space. And notice, we used a lot of different tools and things that we had developed over the course of this-- over the course of this course so far.

I keep using this word completion even though we haven't really talked about it. I kind of left it out of the first chapter because I wasn't going to use the second chapter from the lecture notes that are usually used for this course.

But from this fact-- so you should think of the completion as the smallest Banach space containing a certain norm space.

But what this statement here along with the theorem due to Riesz and Fischer is that the completion of continuous functions over a, b with norm given by the Lp norm, which is, in fact, a norm on continuous functions because if a continuous function is equal to 0 almost everywhere, it has to be 0. So what this proves is that the completion of the continuous functions with this norm is equal to the space Lp.

So now we're going to move on back to some more general theory of functional analysis. This was specific to measure and integration on the real numbers. Now we're going to go to more general topics and probably more intuitive topics because a lot of it has analogs.

I mean, a lot of functional analysis has-- it's supposed to be in some way analogous to stuff you've seen in linear algebra. Some of it definitely is not. For example, in one of the assignments, you proved that the unit ball and little Lp is not compact for-- over the natural numbers, the set of sequences that have little-- finite little Lp norm. That that's not compact.

While from calculus, that in Rn, the unit ball is compact. That's the Burrill theorem or Bolzano-Weierstrass depending on if you take as your starting definition of compactness in terms of open sets or in terms of sequences having a convergent subsequence, which are equivalent for metric-- at least in a metric space.

OK. Now here we're going to move on to the topic of Hilbert spaces. So these are special in the sense that the norm comes from an inner product-- maybe you saw an inner product in linear algebra. Or you should have at least.

And therefore, you have notions of being orthogonal, you have notions of projections, and these-- we saw things that have that flavor when we're talking about Banach spaces in general and we were talking about modding out by a subspace, and you can think of the equivalence class corresponding to an element as the projection onto the complement of that subspace, but it wasn't an exact analogy.

But a lot of exact analogies will now occur for Hilbert spaces, and then certain operators on Hilbert spaces will be very analogous to self-adjoint matrices or symmetric matrices which you saw in linear algebra.

And of course, from a applied standpoint-- I should say applied-- Hilbert spaces, this is where the action-- this is where-- the setting of quantum mechanics. Quantum mechanics takes place in a Hilbert space.

The elements are square integrable, now that we've dealt with that, functions over R3, if we're in three dimensions or if we're on the line R, that have L2 norm equal to 1, along with the Schrodinger equation.

So Hilbert spaces are very important. They arise naturally in many problems. And because of this additional structure of them-- of the norm coming from an inner product, you can say a lot more things about them.

So before we get to Hilbert spaces, let me add a "pre" before that. So pre-Hilbert spaces. So I said that Hilbert spaces are going to be Banach spaces that come from a norm. Pre-Hilbert spaces, these are just norm spaces that come from an inner product. So make the following definition.

A pre-Hilbert space H, this is a vector space. Typically over C, but you can also just take it over R, that's fine as well. But I'll just for definiteness say over C. With a Hermitian inner product.

So this is new terminology, maybe this is new terminology, too, so let me write out what it means-- what is a Hermitian inner product. So this is, i.e., a map, usually denoted using brackets from H-cross-H into the complex numbers.

Satisfying certain properties-- so such that-- so one, it is linear and the first variable-- so for all lambda 1, lambda 2, v1, v2, if I look at the inner product of v1, lambda 1 times v1 plus lambda 2 times v2 with w, this is equal to lambda 1 times v1 times the inner product with w plus lambda 2 v2 inner product with w.

For all V, w, and-- I wrote capital V, I should have-- written H a minute ago. Now we're in Hilbert spaces, pre-Hilbert spaces. For all v and w, the inner product of v and w is equal to the complex conjugate of the inner product of w with v.

And the following, which is positive definiteness of the inner product, if to call it that, for all v and H, m the inner product would be with itself, this is bigger than or equal to 0, so it's a real number and it's non-negative. And this one, it equals 0 if and only if v is a zero element.

OK. So let me make a couple of remarks. First off, let's say this is remark 1. The third quantity does imply that the only thing orthogonal to everything in the space is a zero element. So v is in H and v is orthogonal to everything-- I keep using the word orthogonal. I should just the inner product with zero.

It gives me 0 for every element, this implies that v equals 0, and of course, the converse applies, too, that v is equal to 0, then the inner product with 0 and every element is 0 just by linearity.

Two is that if I have two elements v and w and a scalar lambda, then this is equal to the complex conjugate of lambda times w with v, and by the first property, the lambda pops out, and therefore, if I-- complex conjugate of a product is a product of the complex conjugates. And then if I undo this, I can get this.

So it's linear in the first entry, meaning the constants just come out. But if I have a constant in the second-- or a scalar multiple in the second entry, then that comes out as well, but with a complex conjugate over it. That's all I wanted to say.

So I said that pre Hilbert spaces are-- you naturally think of them as norm spaces where the norm comes from the inner product. Here, we have in a product. Where's the norm? So definition. If H is a pre-Hilbert space with inner product denoted as before, we define using this--

So I'm not calling it a norm yet. I'm just saying we define this function on H to be in a product v with itself raised to the 1/2 power. In the end or in a minute we'll show that this is, in fact, a norm.

But for now I'm just going to call it this function on H or possible norm on H. All right, so we have the following theorem. So this is valid in any pre-Hilbert space. For all u, v, and H of pre-Hilbert space, if I take the absolute value of the inner product of u and v, this is less than or equal to this norm-looking thing of u times the norm-looking thing of v.

So this shouldn't come as a complete surprise. Right down-- if we took H to be Rn, and so then now this is a vector space over R, then and the inner product would just be the dot product. This is stating the Cauchy-Schwarz inequality that you know and love from before.

So what's the proof? Let's let f of t be the norm of u plus t times v. So I said norm, but I haven't proved it's a norm yet, so you're going to have to forgive if I keep calling it a norm, but in the end it is.

Let f of t be this thing squared, which, we note, it's a non-negative number. Because the product of v with v is a non-negative number taking that to the 1/2 power, so this is u plus tv inner product u plus tv which is non-negative.

Now if we compute all this out using how the linearity works for inner products, this-- I get u inner product u plus t squared v inner product v plus t times u inner product v plus t times inner product of v with u.

And this is the complex conjugate of this complex number. So let me just rewrite this-- u squared plus t squared or v squared plus-- so again, this number is the complex conjugate of this number. And when I add a complex number to its complex conjugate, I get twice times the real part. 2t real part of u and v.

Now, this is just a polynomial with a non-negative thing out in front of t squared. So it has a minimum somewhere. And this minimum has to be non-negative since this function is greater than or equal to 0. Is greater than or equal to 0, I should say, just as a sentence.

Now, where does this minimum occur? It occurs where the derivative is equal to 0. Now f prime of t then equals 0 implies-- or if and only if t min is equal to minus the real part. So I will leave this calculus to you.

I mean, this is just a polynomial. Take the derivative with respect to t and solve for when that's equal to 0. And therefore, I get that this minimum-- so f evaluated at this point is non-negative, so 0 is less than or equal to f of t min, which, when you stick that into this here, you get norm of u squared minus real part of u v squared over norm of v squared.

And therefore, the real part of u, v absolute value-- so this-- all of this is bigger than or equal to 0. So if I move this over to that side and multiplied by norm squared, take the square root, I get that this is less than or equal to-- le me put a-- u times norm v.

So this is almost what I want. I want the absolute value of the inner product with u and v. But all I have is the real part of inner product with u and v.

So of course, if the inner product of u and v is 0, this inequality I want to prove is automatic. Also, if u or v is 0, then this inequality I wanted to prove is automatic. So that's why I'm actually just dealing with the non-trivial case.

So if this inner product is 0, then we're already done because the right-hand side is non-negative. So suppose this is non-zero, but lambda be this complex number u, v inner product complex conjugate over the absolute value, then the absolute value of this complex number is equal to 1 because the absolute value of the conjugate is equal to the absolute value of the original complex number.

And lambda times u, v, which is equal to lambda. So I should say u, v, this is equal to lambda times u, v, which is equal to lambda u times v.

Now this is equal to this, and therefore, it's equal to the real part of it. So this is equal to a real number, and therefore it's equal to its real part. We pulled a similar trick when we were talking about the triangle inequality for integral functions that now take values and the complex number-- complex numbers.

So this is equal to the real part of this, which is less than or equal to-- by what we've done here, we've already proven this inequality holds for every u and v, so it also holds for lambda times u times v. And now-- so simple off to the side, that if I take lambda u, an inner product with lambda u, if I take the 1/2 power of that, that gives me this quantity.

But let me just compute this. This is equal to-- lambda comes out from here. And then complex conjugate of lambda comes out from there, and this is equal to-- but this is equal to 1.

So we see that this is equal to this, and therefore, raising it to the 1/2 power is equal to-- this quantity here is equal to this quantity. And we proven what we wanted to do.

OK. And so this was a Cauchy-Schwarz inequality and a general pre-Hilbert space with this quantity that I referred to as a norm but I haven't proved it's a norm yet. Next time we will use the Cauchy-Schwarz inequality to prove that, in fact, this thing that I'm masquerading around in a norm notation is, in fact, a norm on a pre-Hilbert space.

And from there, we'll introduce Hilbert spaces which are those pre-Hilbert spaces with this norm that are actually complete. And really, for any kind of-- so for-- and we'll prove this at some point, there's really only two types of reasonable Hilbert spaces.

And I mean this in a very strong sense, not in a loose sense, that the first type is just finite-dimensional-- so think of C raised to the n. So n tuples of complex numbers where the inner product is definable in a natural way.

Or a little L2 which is these-- this set of sequences that have finite little L2 norm. This is basically the only other type of Hilbert space that there is, and I'll say what I mean by that.

How it'll work is we'll basically show that every separable Hilbert space, which is what I mean by reasonable, that it's the most reasonable spaces we work with are separable, meaning they have countable dense subset, has a countable orthogonal basis, which is what--

Orthonormal basis I haven't defined. It's not a Hamel basis, but it serves the same purpose, meaning you can't write every element as a finite linear combination of orthonormal basis, but you can write it as an infinite expansion in orthonormal basis. Think Fourier series.

And this is what provides this identification of a separable Hilbert space with either a C to the n if this or orthonormal basis is finite, or little L2 if this or orthonormal basis is countably infinite. And we'll get to that possibly by the end of next lecture.