Assignments are to be submitted to Gradescope by 24:00.

The following is necessary for the first two problems.

Theorem. *Let $E \subset \mathbb{R}$ be measurable, and let $h_n \in L^1(E)$, $n \in \mathbb{N}$, such that $\int_E |h_n| \to 0$ as $n \to \infty$. Then there exists a subsequence $\{h_{n_j}\}_j$ such that for almost every $x \in E$, $\lim_{j\to\infty} h_{n_j}(x) = 0$.*

Proof. Let $j \in \mathbb{N}$. Then

$$m(\{x \in E \mid |h_n(x)| \geq 2^{-j}\}) = \int_{\{|h_n|>2^{-j}\}} 1 \leq 2^j \int_{\{|h_n|>2^{-j}\}} |h_n| \leq 2^j \int_E |h_n| \to 0,$$

as $n \to \infty$. Thus, there exists a sequence of integers $n_1 < n_2 < n_3 < \cdots$ such that for all $j \in \mathbb{N}$,

$$m(\{x \in E \mid |h_{n_j}(x)| \geq 2^{-j}\}) \leq 2^{-j}.$$

Let $E_j = \{x \in E \mid |h_{n_j}(x)| \geq 2^{-j}\}$, and $F := \bigcap_{k=1}^\infty \Big(\bigcup_{j=k}^\infty E_j\Big)$. Then $m\Big(\bigcup_{j=1}^\infty E_j\Big) \leq \sum_j m(E_j) \leq \sum_j 2^{-j} = 1 < \infty$, so by continuity of Lebesgue measure

$$\begin{aligned} m(F) &= \lim_{k\to\infty} m\Big(\bigcup_{j=k}^\infty E_j\Big) \\ &\leq \lim_{k\to\infty} \sum_{j=k}^\infty m(E_j) \\ &\leq \lim_{k\to\infty} \sum_{j=k}^\infty 2^{-j} \\ &= \lim_{k\to\infty} 2^{-k+1} = 0. \end{aligned}$$

One can then easily check that for all $x \in E \backslash F = \bigcup_{k=1}^\infty \Big(\bigcap_{j=k}^\infty (E \backslash E_j)\Big)$, we have $\lim_{j\to\infty} |f_{n_j}(x)| = 0$, and thus $\{x \in E \mid \lim_{j\to\infty} |f_{n_j}(x)| \neq 0\} \subset F$ (a set of measure zero) proving the theorem. □

Definition. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that*

- *for all $x \in \mathbb{R}$, $\varphi(-x) = \varphi(x)$ and $\varphi(x) \geq 0$,*
- *if $|x| > 1$ then $\varphi(x) = 0$,*
- *$\int_{\mathbb{R}} \varphi(x)dx = \int_{-1}^1 \varphi(x)dx = 1$.*

For $\epsilon > 0$, we define $\varphi_\epsilon(x) = \frac{1}{\epsilon}\varphi\big(\frac{x}{\epsilon}\big)$, and note $\int_{\mathbb{R}} \varphi_\epsilon(x)dx = 1$. The family of functions $\{\varphi_\epsilon\}$ is called a a family of mollifiers.

For problems 1 and 2, the arguments used in showing convergence of Fourier series in $L^2([-\pi, \pi])$ might be useful.

1. Let $g \in C(\mathbb{R})$ with the property that there exists $R > 0$ such that if $|x| > R$ then $g(x) = 0$. Note that this implies g is uniformly continuous on $\mathbb{R}$. For $\epsilon > 0$, define

$$g_\epsilon(x) := \int_{\mathbb{R}} \varphi_\epsilon(x-y)g(y)dy = \int_{\mathbb{R}} \varphi_\epsilon(z)g(x-z)dz.$$

   (a) Prove that g_ϵ is infinitely differentiable and if $|x| > R + \epsilon$ then $g_\epsilon(x) = 0$.

   (b) Prove that $\|g_\epsilon - g\|_\infty \to 0$ as $\epsilon \to 0^+$.

   (c) Let $f \in L^2(\mathbb{R})$ and $\delta > 0$. Prove that there exists an infinitely differentiable function h with the property that there exists $S > 0$ such that if $|x| > S$ then $h(x) = 0$ and

$$\|f - h\|_2 < \delta.$$

   This shows that the subspace of smooth, compactly supported functions is dense in $L^2(\mathbb{R})$.

2. For $f \in L^2(\mathbb{R})$ and $\epsilon > 0$, we define

$$f_\epsilon(x) = \int_{\mathbb{R}} \varphi_\epsilon(x-y)f(y)dy.$$

   (a) Prove that for all $f \in L^2(\mathbb{R})$, $\|f_\epsilon\|_2 \le \|f\|_2$.
   Hint: First prove the inequality for all continuous functions in $L^2(\mathbb{R})$. Now let $f \in L^2(\mathbb{R})$ and $f_n \in C(\mathbb{R}) \cap L^2(\mathbb{R})$, $n \in \mathbb{N}$, such that $\|f_n - f\|_2 \to 0$. By the estimate for continuous functions, the sequence $\{(f_n)_\epsilon\}_n$ is a Cauchy sequence in $L^2(\mathbb{R})$, so there exists $g \in L^2(\mathbb{R})$ such that $\|(f_n)_\epsilon - g\|_2 \to 0$ as $n \to \infty$. Prove that for all $x \in \mathbb{R}$, $|(f_n)_\epsilon(x) - f_\epsilon(x)| \to 0$ as $n \to \infty$, and use the Theorem above to conclude $g = f_\epsilon$. Finally, conclude $\|f_\epsilon\|_2 \le \|f\|_2$.

   (b) Prove that for all $f \in L^2(\mathbb{R})$, $\|f_\epsilon - f\|_2 \to 0$ as $\epsilon \to 0^+$.

3. Let H be a Hilbert space. Suppose that $P \in \mathcal{B}(H,H)$ is a projection and for all $u, v \in H$, $\langle Pu, v\rangle = \langle u, Pv\rangle$. Prove that $W := \text{Range}(P) = \{Pu \mid u \in H\}$ is a closed subspace of H and $P = \Pi_W$.

4. Let H be a Hilbert space, and let $C \subset H$ be a closed convex subset.

   (a) Prove that for all $u \in H$ there exists a unique element $v \in C$ such that

$$\|u - v\| = \inf_{w \in C} \|u - w\|.$$

   We denote this element in C by $v = P_C u$.

   (b) Let $u \in H$ and $v \in C$. Prove that $v = P_C u$ if and only if for all $w \in C$

$$\text{Re}\langle u - v, w - v\rangle \le 0. \tag{$\dagger$}$$

   Hint: To show that $v = P_C u$ implies ($\dagger$), let $w \in C$, and for $t \in [0,1]$, $w(t) = (1-t)v + tw \in C$. Then

$$\|u - v\|^2 \le \|u - w(t)\|^2, \quad t \in [0,1].$$

   Now expand this out, cancel terms and send $t \to 0^+$.

5. Let $\{h_k\}_k$ be a sequence of non-negative real numbers.

   (a) Prove that $C = \{\{b_k\}_k \in \ell^2 \mid |b_k| \leq h_k \text{ for all } k\}$ is a closed convex subset of ℓ^2.

   (b) For $a = \{a_k\}_k \in \ell^2$, compute $P_C a = \{(P_C a)_k\}_k$.