February 25, 2021

Last time, we proved the Uniform Boundedness Theorem from the Baire Category Theorem, and we'll continue to prove some "theorems with names" in functional analysis today.

Theorem 37 (Open Mapping Theorem)
Let B_1, B_2 be two Banach spaces, and let $T \in \mathcal{B}(B_1, B_2)$ be a surjective linear operator. Then T is an **open map**, meaning that for all open subsets $U \subset B_1$, $T(U)$ is open in B_2.

Proof. We'll begin by proving a specialized result: we'll show that the image of the open ball $B_1(0, 1) = \{b \in B_1 : ||b|| < 1\}$ contains an open ball in B_2 centered at 0. (Then we'll use linearity to shift and scale these balls accordingly.)

Because T is surjective, everything in B_2 is mapped onto, meaning that

$$B_2 = \bigcup_{n \in \mathbb{N}} \overline{T(B(0, n))}$$

(because any element of B_1 is at a finite distance from 0, it must be contained in one of the balls). Now we've written B_2 as a union of closed sets, so by Baire, there exists some $n_0 \in \mathbb{N}$ such that $\overline{T(B(0, n_0))}$ contains an open ball. But T is a linear operator, so this is the same set as $n_0 \overline{T(B(0, 1))}$ (we can check that closure respects scaling and so on). So we have an open ball inside $\overline{T(B(0, 1))}$ – restated, there exists some point $v_0 \in B_2$ and some radius $r > 0$ such that $B(v_0, 4r)$ is contained in $\overline{T(B(0, 1))}$ (the choice of 4 will make arithmetic easier later).

And we want a point that's actually in the image of $B(0, 1)$ (not just the closure), so we pick a point $v_1 = Tu_1 \in T(B(0, 1))$ such that $||v_0 - v_1|| < 2r$. (The idea here is that points in the closure of $T(B(0, 1))$ are arbitrarily close to points actually in $T(B(0, 1))$.) Now $B(v_1, 2r)$ is entirely contained in $B(v_0, 4r)$, which is contained in $\overline{T(B(0, 1)}$, and now we'll show that this closure contains an open ball **centered at 0** (which is pretty close to what we want). For any $||v|| < r$, we have

$$\frac{1}{2}(2v + v_1) \in \frac{1}{2}B(v_1, 2r) \subset \frac{1}{2}\overline{T}(B(0, 1)) = \overline{T(B(0, \tfrac{1}{2}))},$$

and thus $v = -T\left(\frac{u_1}{2}\right) + \frac{1}{2}(2v + v_1)$ is an element of $-T\left(\frac{u_1}{2}\right) + \overline{T(B(0, \frac{1}{2}))}$ (this is not an equivalence class – it's the set of elements $\overline{T(B(0, \frac{1}{2}))}$ all shifted by $-T\left(\frac{u_1}{2}\right)$), and now by linearity this means that our element v must be in the set $\overline{T\left(-\frac{u_1}{2} + B\left(0, \frac{1}{2}\right)\right)}$. But we chose u_1 to have norm less than 1, so $-\frac{u_1}{2}$ and any element of $B(0, \frac{1}{2})$ must both have norm at most $\frac{1}{2}$ (and their sum has norm at most 1). Thus, this set must be contained in $\overline{T(B(0, 1))}$, and therefore the ball of radius r, $B(0, r)$ (in B_2) is contained in $\overline{T(B(0, 1))}$.

But by scaling, we find that $B(0, 2^{-n}r) = 2^{-n}B(0, r)$ is contained in $2^{-n}\overline{T(B(0, 1))} = \overline{T(B(0, 2^{-n}))}$ (repeatedly using homogeneity), and now we'll use that fact to prove that $B(0, \frac{r}{2})$ is contained in $T(B(0, 1))$ (finally removing the closure and proving the specialized result). To do that, take some $||v|| < \frac{r}{2}$; we know that (plugging in $n = 1$) $v \in \overline{T(B(0, \frac{1}{2}))}$. So there exists some $b_1 \in B(0, \frac{1}{2})$ in B_1 such that $||v - Tb_1|| < \frac{r}{4}$ (this is the same idea as above that points in the closure are arbitrarily close to points in the actual set). Then taking $n = 2$, we know that $v - Tb_1 \in \overline{T(B(0, \frac{1}{4}))}$, so there is some $b_2 \in B(0, \frac{1}{4})$ such that $||v - Tb_1 - Tb_2|| < \frac{r}{8}$. Continue iterating this for larger and larger n, so that we have a sequence $\{b_k\}$ of elements in B_1 such that $||b_k|| < 2^{-k}$ and

$$\left|\left| v - \sum_{k=1}^{n} Tb_k \right|\right| < 2^{-n-1}r.$$

And now the series $\sum_{n=1}^{\infty} b_k$ is absolutely summable, and because B_1 is a Banach space, that means that the series is summable, and we have $b \in B_1$ such that $b = \sum_{k=1}^{\infty} b_k$. And

$$||b|| = \lim_{n\to\infty} \left|\left|\sum_{k=1}^{n} b_k\right|\right| \le \lim_{n\to\infty} \sum_{k=1}^{n} ||b_k||$$

by the triangle inequality, and then we can bound this as

$$= \sum_{k=1}^{\infty} ||b_k|| < \sum_{k=1}^{\infty} 2^{-k} = 1.$$

Furthermore, because T is a (bounded, thus) continuous operator,

$$Tb = \lim_{n\to\infty} T\left(\sum_{k=1}^{n} b_k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} Tb_k = v,$$

because we chose our b_k so that $||v - Tb_1 - Tb_2 - \cdots - Tb_k||$ converges to 0. Therefore, since $b \in B(0,1)$, $v \in T(B(0,1))$, and that means the ball $B(0, \frac{r}{2})$ in B_2 is indeed contained in $T(B(0,1))$.

We've basically shown now that 0 remains an interior point if it started as one, and now we'll finish with some translation arguments: if a set $U \subset B_1$ is open, and $b_2 = Tb_1$ is some arbitrary point in $T(U)$, then (by openness of U) there exists some $\varepsilon > 0$ such that $b_1 + B(0,\varepsilon) = B(b_1, \varepsilon)$ is contained in U. Furthermore, by our work above, there exists some δ so that $B(0,\delta) \subset T(B(0,1))$. So this means that

$$B(b_2, \varepsilon\delta) = b_2 + \varepsilon B(0,\delta) \subset b_2 + \varepsilon T(B(0,1)) = T(b_1) + \varepsilon T(B(0,1)) = T(b_1 + B(0,\varepsilon)).$$

But $b_1 + B(0,\varepsilon)$ is contained in U, so indeed we've found a ball around our arbitrary b_2 contained in $T(U)$, and this proves the desired result. □

Corollary 38

If B_1, B_2 are two Banach spaces, and $T \in \mathcal{B}(B_1, B_2)$ is a bijective map, then T^{-1} is in $\mathcal{B}(B_2, B_1)$.

Proof. We know that T^{-1} is continuous if and only if for all open $U \subset B_1$, the inverse image of U by T^{-1} (which is $T(U)$) is open. And this is true by the Open Mapping Theorem. □

From the Open Mapping Theorem, we get this an almost topological result, which gives sufficient conditions for continuity of a linear operator. But first we need to state another result:

Proposition 39

If B_1, B_2 are Banach spaces, then $B_1 \times B_2$ (with operations done entry by entry) with norm

$$||(b_1, b_2)|| = ||b_1|| + ||b_2||$$

is a Banach space.

(This proof is left as an exercise: we just need to check all of the definitions, and a Cauchy sequence in $B_1 \times B_2$ will consist of a Cauchy sequence in each of the individual spaces B_1 and B_2. So it's kind of similar to proving completeness of $\mathbb{R}^2$.)

Theorem 40 (Closed Graph Theorem)
Let B_1, B_2 be two Banach spaces, and let $T : B_1 \to B_2$ be a (not necessarily bounded) linear operator. Then $T \in \mathcal{B}(B_1, B_2)$ if and only if the **graph** of T, defined as

$$\Gamma(T) = \{(u, Tu) : u \in B_1\},$$

is closed in $B_1 \times B_2$.

This can sometimes be easier or more convenient to check than the boundedness criterion for continuity. And normally, proving continuity means that we need to show that for a sequence $\{u_n\}$ converging to u, Tu_n converges and is also equal to Tu. But the Closed Graph Theorem eliminates one of the steps – proving that the graph is closed means that given a sequence $u_n \to u$ **and** a sequence $Tu_n \to v$, we must show that $v = Tu$ (in other words, we just need to show that the convergence point is correct, without explicitly constructing one)!

Proof. For the forward direction, suppose that T is a bounded linear operator (and thus continuous). Then if (u_n, Tu_n) is a sequence in $\Gamma(T)$ with $u_n \to u$ and $Tu_n \to v$, we need to show that (u, v) is in the graph. But

$$v = \lim_{n\to\infty} Tu_n = T\left(\lim_{n\to\infty} u_n\right) = Tu,$$

and thus (u, v) is in the graph and we've proven closedness.

For the other direction, consider the following commutative diagram:

$$\begin{tikzcd} & \Gamma(T) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ B_1 \arrow[rr, "T"] & & B_2 \end{tikzcd}$$

Here, π_1 and π_2 are the projection maps from the graph down to B_1 and B_2 (meaning that $\pi_1(u, Tu) = u$ and $\pi_2(u, Tu) = Tu$). We want to construct a map $S : B_1 \to \Gamma(T)$ (so that $T = \pi_2 \circ S$), and we do so as follows. Since $\Gamma(T)$ is (by assumption) a closed subspace of $B_1 \times B_2$, which is a Banach space, $\Gamma(T)$ must be a Banach space as well. And now π_1, π_2 are continuous maps from the Banach space $\Gamma(T)$ to B_1, B_2 respectively, so π_1 is a bounded linear operator in $\mathcal{B}(\Gamma(T), B_1)$, and similarly $\pi_2 \in \mathcal{B}(\Gamma(T), B_2)$ (we can see this through the calculation $||\pi_2(u, v)|| = ||v|| \le ||u|| + ||v|| = ||(u, v)||$, for example). Furthermore, $\pi_1 : \Gamma(T) \to B_1$ is actually **bijective** (because there is exactly one point in the graph for each u), so by Corollary 38, it has an inverse $S : B_1 \to \Gamma(T)$ which is a bounded linear operator.

And now $T = \pi_2 \circ S$ is the composition of two bounded linear operators, so it is also a bounded linear operator. $\square$

Remark 41. *The Open Mapping Theorem implies the Closed Graph Theorem, but we can also show the converse (so the two are logically equivalent).*

Each of the results so far has been trying to answer a question, and our next result, the **Hahn-Banach Theorem**, is asking whether the dual space of a general nontrivial normed space is trivial. (In other words, we want to know whether there are any normed spaces whose space of functionals $\mathcal{B}(V, \mathbb{K})$ only contains the zero function.) For example, we mentioned that for any finite $p \ge 1$, ℓ^p and ℓ^q are dual is $\frac{1}{p} + \frac{1}{q} = 1$, and it's also true that $(c_0)' = \ell^1$. So Hahn-Banach will tell us that the dual space has "a lot of elements," but first we'll need an intermediate result from set theory:

Definition 42

A **partial order** on a set E is a relation $\preceq$ on E with the following properties:

- For all $e \in E$, $e \preceq e$.
- For all $e, f \in E$, if $e \preceq f$ and $f \preceq e$, then $e = f$.
- For all $e, f, g \in E$, if $e \preceq f$ and $f \preceq g$, then $e \preceq g$.

An **upper bound** of a set $D \subset E$ is an element $e \in E$ such that $d \preceq e$ for all $d \in D$, and a **maximal element** of E is an element e such that for any $f \in E$, $e \preceq f \implies e = f$ (**minimal element** is defined similarly).

Notably, we do not need to have either $e \preceq f$ or $f \preceq e$ in a partial ordering, and a maximal element does not need to sit "on top" of everything else in E, because we can have other elements "to the side:"

Example 43

If S is a set, we can define a partial order on the powerset of S, in which $E \preceq F$ if E is a subset of F. Then not all sets can be compared (specifically, it doesn't need to be true that either $E \preceq F$ or $F \preceq E$).

Definition 44

Let $(E, \preceq)$ be a partially ordered set. Then a set $C \subset E$ is a **chain** if for all $e, f \in C$, we have either $e \preceq f$ or $f \preceq e$.

(In other words, we can always compare all elements in a chain.)

Proposition 45 (Zorn's lemma)

If every chain in a nonempty partially ordered set E has an upper bound, then E contains a maximal element.

We'll take this as an **axiom of set theory**, and we'll give an application of this next lecture. But we can use it to prove other things as well, like the **Axiom of Choice**.

Definition 46

Let V be a vector space. A **Hamel basis** $H \subset V$ is a linearly independent set such that every element of V is a finite linear combination of elements of H.

We know from linear algebra that we find a basis and calculate its cardinality to find the dimension for finite-dimensional vector spaces. (So a Hamel basis for $\mathbb{R}^n$ can be the standard n basis elements, and a Hamel basis for ℓ^1 can be $(1, 0, 0, \cdots), (0, 1, 0, \cdots)$, and so on.) And next time, we'll use Zorn's lemma to talk more about these Hamel bases!

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