18.102 Introduction to Functional Analysis
Spring 2009

PATRIOT PROBLEMS FOR 18.102, SPRING 2009
DON'T HAND THEM IN!.

RICHARD MELROSE

Here I suggest that for fun over the Patriot's Day long weekend you work your way through the theory of Hilbert-Schmidt operators on a separable, infinite-dimensional, Hilbert space.

Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *Hilbert-Schmidt* (really 'with respect to this orthonormal basis' but see below) if

$$\|T\|^2_{\mathrm{HS}} = \sum_{i=1}^{\infty} \|Te_i\|^2 < \infty. \tag{9.1}$$

(1) Show that a finite rank operator is Hilbert-Schmidt.
(2) Show that the Hilbert-Schmidt operators form a linear space.
(3) Let $\{f_i\}$ be another (or of course even the same) orthonormal basis. Use the expansion of the norm to see that

$$\|T\|^2_{\mathrm{HS}} = \sum_{i,j=1}^{\infty} |(Te_i, f_j)|^2. \tag{9.2}$$

(4) Use the preceding identity to see that

$$\|T\|^2_{\mathrm{HS}} = \sum_{j=1}^{\infty} \|T^* f_j\|^2. \tag{9.3}$$

(5) Applying this conclusion twice check that the sum on the right in (9.1) is independent of the orthonormal basis used to define it and hence $\mathrm{HS}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ is a well-defined subspace.
(6) Show that $T \in \mathrm{HS}(\mathcal{H}) \Longrightarrow T^* \in \mathrm{HS}(\mathcal{H})$.
(7) Show that $\mathrm{HS}(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$ consists of compact operators. (Hint: Finite rank approximation is one approach that works).
(8) Show, directly from the original definition, that if $B \in \mathcal{B}(\mathcal{H})$ and $T \in \mathrm{HS}(\mathcal{H})$ then $BT \in \mathrm{HS}(\mathcal{H})$.
(9) Using the results above show that $\mathrm{HS}(\mathcal{H})$ is an ideal, meaning that $B_1TB_2 \in \mathrm{HS}(\mathcal{H})$ if $T \in \mathrm{HS}(\mathcal{H})$ and B_1, $B_2 \in \mathcal{B}(\mathcal{H})$.
(10) Show that $\mathrm{HS}(\mathcal{H})$ is a Hilbert space with an inner product which is compatible with the norm $\|\cdot\|_{\mathrm{HS}}$.

Department of Mathematics, Massachusetts Institute of Technology