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18.102 Introduction to Functional Analysis
Spring 2009

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**PROBLEM SET 3 FOR 18.102, SPRING 2009
DUE 11AM TUESDAY 3 MAR.**

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This problem set is also intended to be a guide to what will be on the in-class test on March 5. In particular I will ask you to prove some of the properties of the Lebesgue integral, as below, plus one more abstract proof. Recall that equality a.e. (almost everywhere) means equality on the complement of a set of measure zero.

1. PROBLEM 3.1

If f and $g \in \mathcal{L}^1(\mathbb{R})$ are Lebesgue integrable functions on the line show that

- (1) If $f(x) \geq 0$ a.e. then $\int f \geq 0$.
- (2) If $f(x) \leq g(x)$ a.e. then $\int f \leq \int g$.
- (3) If f is complex valued then its real part, $\operatorname{Re} f$, is Lebesgue integrable and $|\int \operatorname{Re} f| \leq \int |f|$.
- (4) For a general complex-valued Lebesgue integrable function

$$(1) \quad \left| \int f \right| \leq \int |f|.$$

Hint: You can look up a proof of this easily enough, but the usual trick is to choose $\theta \in [0, 2\pi)$ so that $e^{i\theta} \int f = \int (e^{i\theta} f) \geq 0$. Then apply the preceding estimate to $g = e^{i\theta} f$.

- (5) Show that the integral is a continuous linear functional

$$(2) \quad \int : L^1(\mathbb{R}) \longrightarrow \mathbb{C}.$$

2. PROBLEM 3.2

If $I \subset \mathbb{R}$ is an interval, including possibly $(-\infty, a)$ or (a, ∞) , we *define* Lebesgue integrability of a function $f : I \longrightarrow \mathbb{C}$ to mean the Lebesgue integrability of

$$(1) \quad \tilde{f} : \mathbb{R} \longrightarrow \mathbb{C}, \quad \tilde{f}(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I. \end{cases}$$

The integral of f on I is then defined to be

$$(2) \quad \int_I f = \int \tilde{f}.$$

- (1) Show that the space of such integrable functions on I is linear, denote it $\mathcal{L}^1(I)$.
- (2) Show that if f is integrable on I then so is $|f|$.
- (3) Show that if f is integrable on I and $\int_I |f| = 0$ then $f = 0$ a.e. in the sense that $f(x) = 0$ for all $x \in I \setminus E$ where $E \subset I$ is of measure zero as a subset of \mathbb{R} .

- (4) Show that the set of null functions as in the preceding question is a linear space, denote it $\mathcal{N}(I)$.
 (5) Show that $\int_I |f|$ defines a norm on $L^1(I) = \mathcal{L}^1(I)/\mathcal{N}(I)$.
 (6) Show that if $f \in \mathcal{L}^1(\mathbb{R})$ then

$$(3) \quad g : I \longrightarrow \mathbb{C}, \quad g(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I \end{cases}$$

is integrable on I .

- (7) Show that the preceding construction gives a *surjective and continuous* linear map ‘restriction to I ’

$$(4) \quad L^1(\mathbb{R}) \longrightarrow L^1(I).$$

(Notice that these are the quotient spaces of integrable functions modulo equality a.e.)

3. PROBLEM 3.3

Really continuing the previous one.

- (1) Show that if $I = [a, b]$ and $f \in L^1(I)$ then the restriction of f to $I_x = [x, b]$ is an element of $L^1(I_x)$ for all $a \leq x < b$.
 (2) Show that the function

$$(1) \quad F(x) = \int_{I_x} f : [a, b] \longrightarrow \mathbb{C}$$

is continuous.

- (3) Prove that the function $x^{-1} \cos(1/x)$ is not Lebesgue integrable on the interval $(0, 1]$. Hint: Think about it a bit and use what you have shown above.

4. PROBLEM 3.4

[Harder but still doable] Suppose $f \in \mathcal{L}^1(\mathbb{R})$.

- (1) Show that for each $t \in \mathbb{R}$ the translates

$$(1) \quad f_t(x) = f(x - t) : \mathbb{R} \longrightarrow \mathbb{C}$$

are elements of $\mathcal{L}^1(\mathbb{R})$.

- (2) Show that

$$(2) \quad \lim_{t \rightarrow 0} \int |f_t - f| = 0.$$

This is called ‘Continuity in the mean for integrable functions’. Hint: I will add one!

- (3) Conclude that for each $f \in \mathcal{L}^1(\mathbb{R})$ the map (it is a ‘curve’)

$$(3) \quad \mathbb{R} \ni t \longmapsto [f_t] \in L^1(\mathbb{R})$$

is continuous.

5. PROBLEM 3.5

In the last problem set you showed that a continuous function on a compact interval, extended to be zero outside, is Lebesgue integrable. Using this, and the fact that step functions are dense in $L^1(\mathbb{R})$ show that the linear space of continuous functions on \mathbb{R} each of which vanishes outside a compact set (which depends on the function) form a dense subset of $L^1(\mathbb{R})$.

6. PROBLEM 3.6

- (1) If $g : \mathbb{R} \rightarrow \mathbb{C}$ is bounded and continuous and $f \in \mathcal{L}^1(\mathbb{R})$ show that $gf \in \mathcal{L}^1(\mathbb{R})$ and that

$$(1) \quad \int |gf| \leq \sup_{\mathbb{R}} |g| \cdot \int |f|.$$

- (2) Suppose now that $G \in \mathcal{C}([0, 1] \times [0, 1])$ is a continuous function (I use $\mathcal{C}(K)$ to denote the continuous functions on a compact metric space). Recall from the preceding discussion that we have defined $L^1([0, 1])$. Now, using the first part show that if $f \in L^1([0, 1])$ then

$$(2) \quad F(x) = \int_{[0,1]} G(x, \cdot) f(\cdot) \in \mathbb{C}$$

(where \cdot is the variable in which the integral is taken) is well-defined for each $x \in [0, 1]$.

- (3) Show that for each $f \in L^1([0, 1])$, F is a continuous function on $[0, 1]$.

- (4) Show that

$$(3) \quad L^1([0, 1]) \ni f \mapsto F \in \mathcal{C}([0, 1])$$

is a bounded (i.e. continuous) linear map into the Banach space of continuous functions, with supremum norm, on $[0, 1]$.