18.102 Introduction to Functional Analysis
Spring 2009

PROBLEM SET 1 FOR 18.102, SPRING 2009 SOLUTIONS

RICHARD MELROSE

Full marks will be given to anyone who makes a good faith attempt to answer each question. The first four problems concern the 'little L p' spaces l^p. Note that you have the choice of doing everything for $p = 2$ or for all $1 \le p < \infty$.

Everyone who handed in a script received full marks.

1. Problem 1.1

Write out a proof (you can steal it from one of many places but at least write it out in your own hand) either for $p = 2$ or for each p with $1 \le p < \infty$ that

$$l^p = \{a : \mathbb{N} \longrightarrow \mathbb{C}; \sum_{j=1}^{\infty} |a_j|^p < \infty,\ a_j = a(j)\}$$

is a normed space with the norm

$$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{\frac{1}{p}}.$$

This means writing out the proof that this is a linear space and that the three conditions required of a norm hold.

Solution:- We know that the functions from any set with values in a linear space form a linear space – under addition of values (don't feel bad if you wrote this out, it is a good thing to do once). So, to see that l^p is a linear space it suffices to see that it is closed under addition and scalar multiplication. For scalar multiples this is clear:-

$$|ta_i| = |t||a_i| \text{ so } \|ta\|_p = |t|\|a\|_p \tag{1}$$

which is part of what is needed for the proof that $\|\cdot\|_p$ is a norm anyway. The fact that $a,\ b \in l^p$ imples $a + b \in l^p$ follows once we show the triangle inequality or we can be a little cruder and observe that

$$\begin{aligned} |a_i + b_i|^p &\le (2\max(|a|_i, |b_i|))^p = 2^p \max(|a|_i^p, |b_i|^p) \le 2^p(|a_i| + |b_i|) \\ \|a + b\|_p^p &= \sum_j |a_i + b_i|^p \le 2^p(\|a\|^p + \|b\|^p), \end{aligned} \tag{2}$$

where we use the fact that t^p is an increasing function of $t \ge 0$.

Now, to see that l^p is a normed space we need to check that $\|a\|_p$ is indeed a norm. It is non-negative and $\|a\|_p = 0$ implies $a_i = 0$ for all i which is to say $a = 0$. So, only the triangle inequality remains. For $p = 1$ this is a direct consequence of the usual triangle inequality:

$$\|a + b\|_1 = \sum_i |a_i + b_i| \le \sum_i (|a_i| + |b_i|) = \|a\|_1 + \|b\|_1. \tag{3}$$

For $1 < p < \infty$ it is known as Minkowski's inequality. This in turn is deduced from Hölder's inequality – which follows from Young's inequality! The latter says if $1/p + 1/q = 1$, so $q = p/(p-1)$, then

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q} \ \forall \ \alpha, \beta \ge 0. \tag{4}$$

To check it, observe that as a function of $\alpha = x$,

$$f(x) = \frac{x^p}{p} - x\beta + \frac{\beta^q}{q} \tag{5}$$

if non-negative at $x = 0$ and clearly positive when $x >> 0$, since x^p grows faster than $x\beta$. Moreover, it is differentiable and the derivative only vanishes at $x^{p-1} = \beta$, where it must have a global minimum in $x > 0$. At this point $f(x) = 0$ so Young's inequality follows. Now, applying this with $\alpha = |a_i|/\|a\|_p$ and $\beta = |b_i|/\|b\|_q$ (assuming both are non-zero) and summing over i gives Hölder's inequality

$$\begin{gathered} |\sum_i a_i b_i|/\|a\|_p\|b\|_q \le \sum_i |a_i||b_i|/\|a\|_p\|b\|_q \le \sum_i \left(\frac{|a_i|^p}{\|a\|_p^p p} + \frac{|b_i|^q}{\|b\|_q^q q} \right) = 1 \\ \Longrightarrow |\sum_i a_i b_i| \le \|a\|_p\|b\|_q. \end{gathered} \tag{6}$$

Of course, if either $\|a\|_p = 0$ or $\|b\|_q = 0$ this inequality holds anyway.

Now, from this Minkowski's inequality follows. Namely from the ordinary triangle inequality and then Minkowski's inequality (with q power in the first factor)

$$\begin{aligned} \sum_i |a_i + b_i|^p &= \sum_i |a_i + b_i|^{(p-1)}|a_i + b_i| \\ &\le \sum_i |a_i + b_i|^{(p-1)}|a_i| + \sum_i |a_i + b_i|^{(p-1)}|b_i| \\ &\le \left(\sum_i |a_i + b_i|^p \right)^{1/q} (\|a\|_p + \|b\|_q) \end{aligned} \tag{7}$$

gives after division by the first factor on the right

$$\|a + b\|_p \le \|a\|_p + \|b\|_p. \tag{8}$$

Thus, l^p is indeed a normed space.

I did not necessarily expect you to go through the proof of Young-Hölder-Minkowksi, but I think you should do so at some point since I will not do it in class.

2. Problem 1.2

The 'tricky' part in Problem 1.1 is the triangle inequality. Suppose you knew – meaning I tell you – that for each N

$$\left(\sum_{j=1}^N |a_j|^p \right)^{\frac{1}{p}} \quad \text{is a norm on } \mathbb{C}^N$$

would that help?

Solution:- Yes indeed it helps. If we know that for each N

$$\left(\sum_{j=1}^{N}|a_j+b_j|^p\right)^{\frac{1}{p}} \le \left(\sum_{j=1}^{N}|a_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{N}|b_j|^p\right)^{\frac{1}{p}} \tag{1}$$

then for elements of l^p the norms always bounds the right side from above, meaning

$$\left(\sum_{j=1}^{N}|a_j+b_j|^p\right)^{\frac{1}{p}} \le \|a\|_p + \|b\|_p. \tag{2}$$

Since the left side is increasing with N it must converge and be bounded by the right, which is independent of N. That is, the triangle inequality follows. Really this just means it is enough to go through the discussion in the first problem for finite, but arbitrary, N.

3. Problem 1.3

Prove directly that each l^p as defined in Problem 1.1 – or just l^2 – is complete, i.e. it is a Banach space. At the risk of offending some, let me say that this means showing that each Cauchy sequence converges. The problem here is to find the limit of a given Cauchy sequence. Show that for each N the sequence in $\mathbb{C}^N$ obtained by truncating each of the elements at point N is Cauchy with respect to the norm in Problem 1.2 on $\mathbb{C}^N$. Show that this is the same as being Cauchy in $\mathbb{C}^N$ in the usual sense (if you are doing $p=2$ it is already the usual sense) and hence, this cut-off sequence converges. Use this to find a putative limit of the Cauchy sequence and then check that it works.

Solution:- So, suppose we are given a Cauchy sequence $a^{(n)}$ in l^p. Thus, each element is a sequence $\{a_j^{(n)}\}_{j=1}^{\infty}$ in l^p. From the continuity of the norm in Problem 1.5 below, $\|a^{(n)}\|$ must be Cauchy in $\mathbb{R}$ and so converges. In particular the sequence is norm bounded, there exists A such that $\|a^{(n)}\|_p \le A$ for all n. The Cauchy condition itself is that given $\epsilon > 0$ there exists M such that for all $m, n > M$,

$$\|a^{(n)} - a^{(m)}\|_p = \left(\sum_i |a_i^{(n)} - a_i^{(m)}|^p\right)^{\frac{1}{p}} < \epsilon/2. \tag{1}$$

Now for each i, $|a_i^{(n)} - a_i^{(m)}| \le \|a^{(n)} - a^{(m)}\|_p$ so each of the sequences $a_i^{(n)}$ must be Cauchy in $\mathbb{C}$. Since $\mathbb{C}$ is complete

$$\lim_{n\to\infty} a_i^{(n)} = a_i \text{ exists for each } i = 1, 2, \ldots. \tag{2}$$

So, our putative limit is a, the sequence $\{a_i\}_{i=1}^{\infty}$. The boundedness of the norms shows that

$$\sum_{i=1}^{N} |a_i^{(n)}|^p \le A^p \tag{3}$$

and we can pass to the limit here as $n \to \infty$ since there are only finitely many terms. Thus

$$\sum_{i=1}^{N} |a_i|^p \le A^p \ \forall\ N \Longrightarrow \|a\|_p \le A. \tag{4}$$

Thus, $a \in l^p$ as we hoped. Similarly, we can pass to the limit as $m \to \infty$ in the finite inequality which follows from the Cauchy conditions

$$(5) \qquad (\sum_{i=1}^{N} |a_i^{(n)} - a_i^{(m)}|^p)^{\frac{1}{p}} < \epsilon/2$$

to see that for each N

$$(6) \qquad (\sum_{i=1}^{N} |a_i^{(n)} - a_i|^p)^{\frac{1}{p}} \le \epsilon/2$$

and hence

$$(7) \qquad \|a^{(n)} - a\| < \epsilon \ \forall \ n > M.$$

Thus indeed, $a^{(n)} \to a$ in l^p as we were trying to show.

Notice that the trick is to 'back off' to finite sums to avoid any issues of interchanging limits.

4. Problem 1.4

Consider the 'unit sphere' in l^p – where if you want you can set $p = 2$. This is the set of vectors of length 1 :

$$S = \{a \in l^p; \|a\|_p = 1\}.$$

(1) Show that S is closed.
(2) Recall the sequential (so not the open covering definition) characterization of compactness of a set in a metric space (e.g. by checking in Rudin).
(3) Show that S is not compact by considering the sequence in l^p with kth element the sequence which is all zeros except for a 1 in the kth slot. Note that the main problem is not to get yourself confused about sequences of sequences!

Solution:- By the next problem, the norm is continuous as a function, so

$$(1) \qquad S = \{a; \|a\| = 1\}$$

is the inverse image of the closed subset $\{1\}$, hence closed.

Now, the standard result on metric spaces is that a subset is compact if and only if every sequence with values in the subset has a convergent subsequence with limit in the subset (if you drop the last condition then the closure is compact).

In this case we consider the sequence (of sequences)

$$(2) \qquad a_i^{(n)} = \begin{cases} 0 & i \ne n \\ 1 & i = n \end{cases}.$$

This has the property that $\|a^{(n)} - a^{(m)}\|_p = 2^{\frac{1}{p}}$ whenever $n \ne m$. Thus, it cannot have any Cauchy subsequence, and hence cannot have a convergent subsequence, so S is not compact.

This is important. In fact it is a major difference between finite-dimensional and infinite-dimensional normed spaces. In the latter case the unit sphere cannot be compact whereas in the former it is.

5. Problem 1.5

Show that the norm on any normed space is continuous.

Solution:- Right, so I should have put this problem earlier!

The triangle inequality shows that for any u, v in a normed space

$$\|u\| \le \|u - v\| + \|v\|, \ \|v\| \le \|u - v\| + \|u\| \tag{1}$$

which implies that

$$|\|u\| - \|v\|| \le \|u - v\|. \tag{2}$$

This shows that $\| \cdot \|$ is continuous, indeed it is Lipschitz continuous.

Department of Mathematics, Massachusetts Institute of Technology