18.102 Introduction to Functional Analysis
Spring 2009

## Lecture 26. Thursday, May 14:Review

Now, there was one final request before I go through a quick review of what we have done. Namely to state and prove the Hahn-Banach Theorem. This is about extension of functionals. Stately starkly, the basic question is: Does a normed space have *any* non-trivial continuous linear functionals on it? That is, is the dual space always non-trivial (of course there is always the zero linear functional but that is not very amusing). We did not really encounter this problem since for a Hilbert space, or even a pre-Hilbert space, there is always the space itsefl, giving continuous linear functionals through the pairing – Riesz' Theorem says that in the case of a Hilbert space that is all there is. I could have used the Hahn-Banach Theorem to show that any normed space has a completion, but I gave a more direct argument for this, which was in any case much more relevant for the cases of $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ for which we wanted *concrete* completions.

**Theorem 19** (Hahn-Banach)**.** *If $M \subset V$ is a linear subspace of a normed space and $u : M \longrightarrow \mathbb{C}$ is a linear map such that*

$$|u(t)| \le C\|t\|_V \ \forall\ t \in M \tag{26.1}$$

*then there exists a bounded linear functional $U : V \longrightarrow \mathbb{C}$ with $\|U\| \le C$ and $U\big|_M = u$.*

First, by computation, we show that we can extend any continuous linear functional 'a little bit' without increasing the norm.

**Lemma 20.** *Suppose $M \subset V$ is a subspace of a normed linear space, $x \notin M$ and $u : M \longrightarrow \mathbb{C}$ is a bounded linear functional as in* (26.1) *then there exists $u' : M' = \{t' \in V; t' = t + ax,\ a \in \mathbb{C}$ such that*

$$u'\big|_M = u,\ |u'(t + ax)| \le C\|t + ax\|_V,\ \forall\ t \in M,\ a \in \mathbb{C}. \tag{26.2}$$

*Proof.* Note that the decompositon $t' = t + ax$ of a point in $M'$ is unique, since $t + ax = \tilde{t} + \tilde{a}x$ implies $(a - \tilde{a})x \in M$ so $a = \tilde{a}$, since $x \notin M$ and hence $t = \tilde{t}$ as well. Thus

$$u'(t + ax) = u'(t) + au(x) = u(t) + \lambda a,\ \lambda = u'(x) \tag{26.3}$$

and all we have at our disposal is the choice of $\lambda$. Any choice will give a linear functional extending $u$, the problem of course is to arrange the continuity estimate without increasing the constant. In fact if $C = 0$ then $u = 0$ and we can take the zero extension. So we might as well assume that $C = 1$ since dividing $u$ by $C$ arranges this and if $u'$ extends $u/C$ then $Cu'$ extends $u$ and the norm estimate in (26.2) follows. So we are assuming that

$$|u(t)| \le \|t\|_V \ \forall\ t \in M. \tag{26.4}$$

We want to choose $\lambda$ so that

$$|u(t) + a\lambda| \le \|t + ax\|_V \ \forall\ t \in M,\ a \in \mathbb{C}. \tag{26.5}$$

Certainly when $a = 0$ this represents no restriction on $\lambda$. For $a \ne 0$ we can divide through by $a$ and (26.5) becomes

$$|a||u(\frac{t}{a}) - \lambda| = |u(t) + a\lambda| \le \|t + ax\|_V = |a|\|\frac{t}{a} - x\|_V \tag{26.6}$$

and since $t/a \in M$ we only need to arrange that

$$|u(t) - \lambda| \le \|t - x\|_V \ \forall\ u \in M \tag{26.7}$$

and the general case follows.

So, we will choose $\lambda$ to be real. A complex linear functional such as $u$ can be recovered from its real part, so set

$$w(t) = \mathrm{Re}(u(t)) \ \forall\ t \in M \tag{26.8}$$

and just try to extend $w$ to a real functional – it is not linear over the complex numbers of course, just over the reals, but what we want is the anaogue of (26.7):

$$|w(t) - \lambda| \le \|t - x\|_V \ \forall\ t \in M \tag{26.9}$$

which does not involve linearity. What we know about $w$ is the norm estimate (26.4) which implies

$$|w(t_1) - w(t_2)| \le |u(t_1) - u(t_2)| \le \|t_1 - t_2\| \le \|t_1 - x\|_V + \|t_2 - x\|_V. \tag{26.10}$$

Writing this out usual the reality we find

$$\begin{gathered} w(t_1) - w(t_2) \le \|t_1 - x\|_V + \|t_2 - x\|_V \Longrightarrow \\ w(t_1) - \|t_1 - x\| \le w(t_2) + \|t_2 - x\|_V \ \forall\ t_1,\ t_2 \in M. \end{gathered} \tag{26.11}$$

We can then take the sup on the right and the inf on the left and choose $\lambda$ in between – namely we have shown that there exists $\lambda \in \mathbb{R}$ with

$$\begin{aligned} w(t) - \|t - x\|_V \le \sup_{t_2 \in M} (w(t_1) - \|t_1 - x\|) \le \lambda \\ \le \inf_{t_2 \in M} (w(t_1) + \|t_1 - x\|) \le w(t) + \|t - x\|_V \ \forall\ t \in M. \end{aligned} \tag{26.12}$$

This in turn implies that

$$-\|t - x\|_V \le -w(t) + \lambda \le \|t - x\|_V \Longrightarrow |w(t)\lambda| \le -\|t - x\|_V \ \forall\ t \in M. \tag{26.13}$$

This is what we wanted – we have extended the real part of $u$ to

$$w'(t + ax) = w(t) - (\mathrm{Re}\, a)\lambda \text{ and } |w'(t + ax)| \le \|t + ax\|_V. \tag{26.14}$$

Now, finally we get the extension of $u$ itself by 'complexifying' – defining

$$u'(t + ax) = w'(t + ax) - iw'(i(t + ax)). \tag{26.15}$$

This is linear over the complex numbers since

$$\begin{aligned} u'(z(t + ax)) = w'(z(t + ax)) - iw'(iz(t + ax) \\ = w'(\mathrm{Re}\, z(t + ax) + i \,\mathrm{Im}\, z(t + ax)) - iw'(i \,\mathrm{Re}\, z(t + ax)) + iw'(\mathrm{Im}\, z(t + ax)) \\ = (\mathrm{Re}\, z + i \,\mathrm{Im}\, z)w'(t + ax) - i(\mathrm{Re}\, z + i \,\mathrm{Im}\, z)(w'(i(t + ax)) = zu'(t + ax). \end{aligned} \tag{26.16}$$

It certainly extends $u$ from $M$ – since the same identity gives $u$ in terms of its real part $w$.

Finally then, to see the norm estimate note that (as we did long ago) there exists a uniqe $\theta \in [0, 2\pi)$ such that

$$\begin{aligned} |u'(t + ax)| = \mathrm{Re}\, e^{i\theta} u'(t + ax) = \mathrm{Re}\, u'(e^{i\theta} t + e^{i\theta} ax) \\ = w'(e^{i\theta} u + e^{i\theta} ax) \le \|e^{i\theta}(t + ax)\|_V = \|t + ax\|_V. \end{aligned} \tag{26.17}$$

This completes the proof of the Lemma. □

*Of Hahn-Banach.* This is an application of Zorn's Lemma. I am not going to get into the derivation of Zorn's Lemma from the Axiom of Choice, but if you believe the latter – and you are advised to do so, at least before lunchtime – you should believe the former.

So, Zorn's Lemma is a statement about partially ordered sets. A partial order on a set $E$ is a subset of $E \times E$, so a relation, where the condition that $(e, f)$ be in the relation is written $e \prec f$ and it must satisfy

$$e \prec e,\ e \prec f \text{ and } f \prec e \Longrightarrow e = f,\ e \prec f \text{ and } f \prec g \Longrightarrow e \prec g. \tag{26.18}$$

So, the missing ingredient between this and an order is that two elements need not be related at all, either way.

A subsets of a partially ordered set inherits the partial order and such a subset is said to be a *chain* if each pair of its elements *is* related one way or the other. An *upper bound* on a subset $D \subset E$ is an element $e \in E$ such that $d \prec e$ for all $d \in D$. A *maximal* element of $E$ is one which is not majorized, that is $e \prec f$, $f \in E$, implies $e = f$.

**Lemma 21** (Zorn)**.** *If every chain in a (non-empty) partially ordered set has an upper bound then the set contains at least one maximal element.*

Now, we are given a functional $u : M \longrightarrow \mathbb{C}$ defined on some linear subspace $M \subset V$ of a normed space where $u$ is bounded with respect to the induced norm on $M$. We apply this to the set $E$ consisting of *all* extensions $(v, N)$ of $u$ with the same norm. That is, $V \supset N \supset M$ must contain $M$, $v\big|_M = u$ and $\|v\|_N = \|u\|_M$. This is certainly non-empty since it contains $(u, M)$ and has the natural partial order that $(v_1, N_1) \prec (v_2, N_2)$ if $N_1 \subset N_2$ and $v_2\big|_{N_1} = v_1$. You can check that this is a partial order.

Let $C$ be a chain in this set of extensions. Thus for any two elements $(v_i, N_1) \in C$, either $(v_1, N_1) \prec (v_2, N_2)$ or the other way around. This means that

$$\tilde{N} = \bigcup \{N; (v, N) \in C \text{ for some } v\} \subset V \tag{26.19}$$

is a linear space. Note that this union need not be countable, or anything like that, but any two elements of $\tilde{N}$ are each in one of the $N$'s and one of these must be contained in the other by the chain condition. Thus each pair of elements of $\tilde{N}$ is actually in a common $N$ and hence so is their linear span. Similarly we can define an extension

$$\tilde{v} : \tilde{N} \longrightarrow \mathbb{C},\ \tilde{v}(x) = v(x) \text{ if } x \in N,\ (v, N) \in C. \tag{26.20}$$

There may be many pairs $(v, N)$ satisfying $x \in N$ for a given $x$ but the chain condition implies that $v(x)$ is the same for all of them. Thus $\tilde{v}$ is well defined, and is clearly also linear, extends $u$ and satisfies the norm condition $|\tilde{v}(x) \le \|u\|_M \|v\|_V$. Thus $(\tilde{v}, \tilde{N})$ is an upper bound for the chain $C$.

So, the set of all extension $E$, with the norm condition, satisfies the hypothesis of Zorn's Lemma, so must – at least in the mornings – have an maximal element $(\tilde{u}, \tilde{M})$. If $\tilde{M} = V$ then we are done. However, in the contary case there exists $x \in V \setminus \tilde{M}$. This means we can apply our little lemma and construct an extension $(u', \tilde{M}')$ of $(\tilde{u}, \tilde{M})$ which is therefore also an element of $E$ and satisfies $(\tilde{u}, \tilde{M}) \prec (u', \tilde{M}')$. This however contradicts the condition that $(\tilde{u}, \tilde{M})$ be maximal, so is forbidden by Zorn. □

There are many applications of Zorn's Lemma, the main one being something like this:-

**Proposition 33.** *For any normed space $V$ and element $x \in V$ there is a continuous linear functional $f : V \longrightarrow \mathbb{C}$ with $f(x) = 1$ and $\|f\| \leq \|x\|_V$.*

*Proof.* Start with the one-dimensional space, $M$, spanned by $x$ and define $u(zx) = z$. This has norm $\|x\|_V$. Extend it and you will get an admissible functional $f$. $\square$

Now, finally the review!