18.102 Introduction to Functional Analysis
Spring 2009

PREPARATORY QUESTIONS FOR TEST 1 FOR 18.102, SPRING 2009.

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These questions, in addition to those in Problems3, are intended to help you study for the test on Thursday March 5. In fact the questions on the test will be very similar to some of those below or on the problem sets.

1. Problem pT1.1

We have defined a set of measure zero $E \subset \mathbb{R}$ by the condition that there exist an absolutely summable series of step functions f_n for which

$$E \subset \{x \in \mathbb{R}; \sum_n |f_n(x)| = \infty\}. \tag{1}$$

Definition 1. A set $E \subset \mathbb{R}$ is said to be of *Lebesgue measure zero* if for each $\delta > 0$ there exists a countable collection of open intervals (a_i, b_i) such that

$$E \subset \bigcup_i (a_i, b_i), \ \sum_i (b_i - a_i) < \delta. \tag{2}$$

This question asks you to prove that these two concepts are the same; on the test I would only ask the first part:-

Proposition 1. *A set is of measure zero if and only if it is of Lebesgue measure zero.*

You can of course proceed on your own or follow these steps.

(1) $\Longleftarrow$

Show that the definition of Lebesgue measure zero is equivalent to the condition that for each n there exist semi-open intervals $I_i^{(n)}[a_i^{(n)}, b_i^{(n)})$ which are disjoint (for each fixed n) and such that

$$E \subset \bigcup_i I_i^{(n)}, \ \sum_i (b_i^{(n)} - a_i^{(n)}) < 2^{-n}. \tag{3}$$

If $f_i^{(n)}$ are the characteristic functions of the intervals $I_i^{(n)}$ in the previous step show that

$$\sum_{i,n} \int |f_i^{(n)}| < \infty. \tag{4}$$

Arrange this double sequence into an absolutely summable series of step functions which diverges on E.

Conclude that if E is of 'Lebesgue measure zero' then it is 'of measure zero' in the earlier sense.

(2) $\Longrightarrow$

If E is of measure zero in our usual sense, let f_n be an absolutely summable series of step functions as in (1).

For each $\delta > 0$ choose $N = N(\delta)$ such that $\sum\limits_{n>N} \int |f_n| < \delta.$

For each fixed δ and hence N, consider the sets which depend on non-negative integers k and j :

$$J_{k,j} = \{x \in \mathbb{R}; \sum_{N<n\le N+j} |f_n(x)| > 2^k\}. \tag{5}$$

Fixing k, or each j this is a finite union of semi-open intervals, since the f_n are step functions. See if you can construct a covering as needed for the Lebesgue measure zero condition from these.

2. Problem pT1.2

Recall our original definition of integrability of a complex-value function on $\mathbb{R}$. Show, directly from the definition, that the real and imaginary parts of an integrable function are integrable.

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