18.100C Lecture 14 Summary

Take a function $f : U \to \mathbb{R}$ defined on a subset $U \subset \mathbb{R}$. Let p be an interior point of U.

Definition 14.1. *f is differentiable at p, with derivative $f'(p) \in \mathbb{R}$, if*

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p} = f'(p).$$

(Occasionally we will also use derivatives $f'(a)$, $f'(b)$ for a function $f : [a, b] \to \mathbb{R}$; those are defined in the same way).

Definition 14.2. *f is differentiable at p, with derivative $f'(p) \in \mathbb{R}$, if one can write*

$$f(x) = f(p) + f'(p)(x - p) + r(x)(x - p),$$

and the function r satisfies $\lim_{x \to p} r(x) = 0$.

Definition 14.3. *f is differentiable at p, with derivative $f'(p) \in \mathbb{R}$, if the following holds. For every $\epsilon > 0$ there is a $\delta > 0$ such that if $|x - p| < \delta$, then*

$$|f(x) - f(p) - f'(p)(x - p)| \leq \epsilon |x - p|.$$

Theorem 14.4. *The three definitions above are equivalent.*

Theorem 14.5. *If f is differentiable at p, it is also continuous at p.*

Example 14.6. *$f(x) = \exp(x)$ is differentiable, and its derivative is $f'(x) = \exp(x)$. (The same argument can be used to prove the familiar formulae for derivatives of* sin *and* cos*).*

Sum, product, and quotient rule (not discussed in class). Here's the chain rule:

Theorem 14.7. *Let $g : U \to \mathbb{R}$ and $f : V \to \mathbb{R}$ be functions, and p an interior point of U, such that $g(u)$ is an interior point of V. Suppose that g is differentiable and p, and f is differentiable at $g(p)$. Then p is an interior point of the domain of definition of $f \circ g$; $f \circ g$ is differentiable at p; and*

$$(f \circ g)'(p) = f'(g(p))g'(p).$$

Theorem 14.8 (Rolle's theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on all of $[a, b]$, differentiable on (a, b), and such that $f(a) = f(b)$. Then there is some $p \in (a, b)$ such that $f'(p) = 0$.*

Theorem 14.9 (Mean Value Theorem). *Let $f : [a, b] \to \mathbb{R}$ be continuous on all of $[a, b]$, and differentiable on (a, b). Then there is some $p \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(p).$$

Theorem 14.10 (Generalized Mean Value Theorem). *Let $f, g : [a, b] \to \mathbb{R}$ be two functions which are continuous on all of $[a, b]$, and differentiable on (a, b). Then there is some $p \in (a, b)$ such that*

$$(f(b) - f(a))g'(p) = (g(b) - g(a))f'(p).$$