18.100B Problem Set 3 Solutions

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1) We begin by defining $d : V \times V \longrightarrow \mathbb{R}$ such that $d(x,y) = \|x-y\|$. Now to show that this function satisfies the definition of a metric. $d(x,y) = \|x-y\| \geq 0$ and

$$d(x,y) = 0 \iff \|x-y\| = 0 \iff x-y=0 \iff x=y$$

So the function is positive definite.

$$d(x,y) = \|x-y\| = \|-1(y-x)\| = |-1|\|y-x\| = \|y-x\| = d(y,x)$$

Thus the function is symmetric. Finally,

$$d(x,z) = \|x-z\| = \|x-y+y-z\| \leq \|x-y\| + \|y-z\| = d(x,y)+d(y,z)$$

So the triangle inequality holds. Therefore d is a metric.

2) Once again we must verify the properties of a metric. We have defined d_1 as

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

Since d is a metric, it only takes nonnegative values, so d_1 cannot be negative. $d_1(x,y)$ is zero exactly when $d(x,y)$ is, so only for $x=y$. Therefore d_1 is positive definite. Since d is symmetric, d_1 obviously inherits this property. Finally, for $x,y,z \in M$

$$\begin{aligned} d_1(x,y)+d_1(y,z) &= \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} = \frac{d(x,y)+d(y,z)+2d(x,y)d(y,z)}{1+d(x,y)+d(y,z)+d(x,y)d(y,z)} \\ &\geq \frac{d(x,y)+d(y,z)+d(x,y)d(y,z)}{1+d(x,y)+d(y,z)+d(x,y)d(y,z)} = 1 - \frac{1}{1+d(x,y)+d(y,z)+d(x,y)d(y,z)} \\ &\geq 1 - \frac{1}{1+d(x,y)+d(y,z)} \geq 1 - \frac{1}{1+d(x,z)} = \frac{d(x,z)}{1+d(x,z)} = d_1(x,z) \end{aligned}$$

So the triangle inequality holds, thus we have a metric. It is easy to see that this metric never takes on a value larger than 1, since $d(x,y) < 1+d(x,y)$, so under the metric d_1, M is bounded.

3) a) $A, B \subseteq M$, M a metric space. Suppose $x \in A^\circ \cup B^\circ$. Without loss of generality, say $x \in A^\circ$. Therefore x is an interior point of A, so $\exists \epsilon > 0$ such that the ball of radius ϵ centered at x is contained in A, or $B_\epsilon(x) \subseteq A$. Since $A \subseteq A \cup B$,

$$B_\epsilon(x) \subseteq A \cup B \Longrightarrow x \in (A\cup B)^\circ$$

This shows that $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$.

b) Now let $x \in A^\circ \cap B^\circ$. Therefore $x \in A^\circ$, so x is an interior point of A, hence $\exists \varepsilon_1 > 0$ such that $B_{\varepsilon_1}(x) \subseteq A$. Similarly, $x \in B^\circ \Longrightarrow \exists \varepsilon_2 > 0$ such that $B_{\varepsilon_2}(x) \subseteq B$. Let $\delta = \min(\varepsilon_1, \varepsilon_2)$. By the triangle inequality,

$$\delta \leq \varepsilon_i \Longrightarrow B_\delta(x) \subseteq B_{\epsilon_i}(x) \Longrightarrow B_\delta(x) \subseteq A \text{ and } B_\delta(x) \subseteq B.$$

Therefore $B_\delta(x) \subseteq A \cap B$, so x is an interior point of $A \cap B$. Hence $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$.

Let $x \in (A \cap B)^\circ$. So $\exists \varepsilon > 0$ with $B_\varepsilon(x) \subseteq A \cap B$. Therefore $B_\varepsilon(x) \subseteq A$ so $x \in A^\circ$, and similarly $x \in B^\circ$. So $x \in A^\circ \cap B^\circ$. Thus $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$. So these two sets are equal.

Let $A = (-1, 0]$ and $B = [0, 1)$. Then 0 is an interior point of neither A nor B, so $0 \notin A^\circ \cup B^\circ$. But $A \cup B = (-1, 1)$, so $0 \in (A \cup B)^\circ$. Therefore in this instance the two sets are unequal.

4) a) If $x \in \partial A$ then every ball around x intersects A and A^c. Thus $x \in A$ and x is a limit point of A^c or $x \in A^c$ and x is a limit point of A. Either way, $x \in \overline{A} \cap \overline{A^c}$, and hence $\partial A \subseteq \overline{A} \cap \overline{A^c}$.

Now let $x \in \overline{A} \cap \overline{A^c}$. Since $x \in \overline{A}$, either $x \in A$ or x is a limit point of A, and in both cases any open ball around x intersects A. Similarly, $x \in \overline{A^c}$ implies any open ball around x intersects A^c. Therefore $x \in \partial A$, so $\overline{A} \cap \overline{A^c} \subseteq \partial A$. So these two sets are equal.

b) Let $p \in \partial A$. By a), $p \in \overline{A}$. Suppose $p \in A^\circ$ then $\exists \varepsilon > 0$ such that $B_\varepsilon(p) \subseteq A$. But this is an open ball centered at p which does not intersect A^c, so $p \notin \partial A$. This contradiction implies that $p \notin A^\circ$.

Now suppose $p \in \overline{A} \setminus A^\circ$. For any $\varepsilon > 0$, $p \in \overline{A}$ gives that $B_\varepsilon(x)$ intersects A, and $p \notin A^\circ$ implies that $B_\varepsilon(x) \nsubseteq A$, so $B_\varepsilon(x)$ intersects A^c. So $p \in \partial A$, and this shows that $\partial A = \overline{A} \setminus A^\circ$.

c) By a), ∂A can be written as the intersection of two closed sets. Thus ∂A is closed.

d) Suppose A is closed. Then $\overline{A} = A$, so by a)

$$\partial A = \overline{A} \cap \overline{A^c} = A \cap \overline{A^c} \subseteq A$$

Converesely, note that for any set B, if $x \notin B$ and $x \notin \partial B$, then there is a positive $r > 0$ such that $B_r(x) \subseteq B^c$ and hence $x \notin \overline{B}$. This implies that

$$\text{for any set } B,\ \overline{B} \subseteq B \cup \partial B.$$

So if $\partial A \subseteq A$, then $\overline{A} \subseteq A \cup \partial A = A \subseteq \overline{A}$ i.e., $A = \overline{A}$ hence A is closed.

5) We will show that $S_r(x) := \{y : d(x, y) = r\}$ is the boundary of $B_r(x)$. It will follow from the previous exercise that

$$\overline{B_r(x)} = \partial B_r(x) \cup B_r(x) = \{y : d(x, y) \leq r\}.$$

It is clear that if y is such that $d(x, y) = r$ then $y \in \partial B_r(x)$ since any ball around y will have points that are closer to x and points that are further away. We just have to show that if $d(x, y) \neq r$, then y is not in $\partial B_r(x)$.

But if $d(x, y) < r$ then for any $0 < \varepsilon < r - d(x, y)$ the ball of radius ε around y is all inside $B_r(x)$ and $y \notin \partial B_r(x)$; and if $d(x, y) > r$ then for any $0 < \delta < d(x, y) - r$ the ball of radius δ around y is all outside of $B_r(x)$ so that again $y \notin \partial B_r(x)$. Thus $\partial B_r(x)$ is precisely $S_r(x)$ and we are done.

Here is an example of a different metric space where this result is not true: Consider $\mathbb{R}^n$ with the discrete metric,

$$\widetilde{d}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

and the ball around any point p with radius 1:

$$B_1(p) = \{q : \widetilde{d}(p,q) < 1\} = \{p\}, \quad \text{while} \quad \{q : \widetilde{d}(p,q) \leq 1\} = \mathbb{R}^n.$$

Notice that the open ball is finite and hence closed. In particular, the closure of $B_1(p)$ is just $\{p\}$ and not $\{q : \widetilde{d}(p,q) \leq 1\}$.

6) We need to show that K is compact or that every open cover of K contains a finite subcover. Let $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ be an open cover of K, so

$$K = \{0, 1, \frac{1}{2}, \dots \frac{1}{n} \dots\} \subseteq \bigcup_{\alpha \in A} \mathcal{U}_\alpha \Longrightarrow \exists \alpha_0 \in A \text{ such that } 0 \in \mathcal{U}_{\alpha_0}$$

Since $\mathcal{U}_{\alpha_0}$ is open, $\exists \varepsilon > 0$ with $B_\varepsilon(0) \subseteq \mathcal{U}_{\alpha_0}$. Because $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $n > N \implies \frac{1}{N} < \varepsilon$. Hence the open set $\mathcal{U}_{\alpha_0}$ contains all of the $\frac{1}{n}$ with $n > N$, i.e., it contains all but finitely many elements of K.

Now, for $i = 1, 2, \dots, N$, $\dfrac{1}{i} \in K$. So $\exists \alpha_i \in A$ such that $\dfrac{1}{i} \in \mathcal{U}_{\alpha_i}$. So we have shown that

$$K \subseteq \bigcup_{i=0}^{N} \mathcal{U}_{\alpha_i},$$

a finite subcover of $\{\mathcal{U}_\alpha\}_{\alpha \in A}$. So every open cover of K contains a finite subcover, which shows that K is compact.

7) We have $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ an open cover of K. Define

$$\mathcal{V}_{\alpha,n} = \{x \in \mathcal{U}_\alpha | B_{\frac{1}{n}}(x) \subseteq \mathcal{U}_\alpha\}^\circ \text{ for all } \alpha \in A, n \in \mathbb{N}.$$

The $\mathcal{U}_\alpha$ are open, so for any point $x \in \mathcal{U}_\alpha$, there is some $n \in \mathbb{N}$ such that

$$B_{\frac{2}{n}}(x) \subseteq \mathcal{U}_\alpha \Longrightarrow B_{\frac{1}{n}}(x) \subseteq \{y \in \mathcal{U}_\alpha | B_{\frac{1}{n}}(y) \subseteq \mathcal{U}_\alpha\} \Longrightarrow x \in \mathcal{V}_{\alpha,n}. \text{ Hence } \bigcup_{n \in \mathbb{N}} \mathcal{V}_{\alpha,n} = \mathcal{U}_\alpha.$$

So taking the union over all $\alpha \in A$, we have

$$\bigcup_{\substack{\alpha \in A \\ n \in \mathbb{N}}} \mathcal{V}_{\alpha,n} = \bigcup_{\alpha \in A} \mathcal{U}_\alpha \supseteq K.$$

So $\{\mathcal{V}_{\alpha,n}\}_{\substack{\alpha \in A \\ n \in \mathbb{N}}}$ is an open cover of K (each set is an interior, thus open). By the compactness of K, there exists a finite subcover $\{\mathcal{V}_{\alpha_i,n_i}\}_{i=1}^{N}$. Let $\delta = (\max_{1 \leq i \leq N} n_i)^{-1}$. Then $\forall x \in K$, $\exists i' \in \{1, 2, \dots, n\}$ with

$$x \in \mathcal{V}_{\alpha_{i'},n_{i'}} \Longrightarrow B_{\frac{1}{n_{i'}}}(x) \subseteq \mathcal{U}_{\alpha_{i'}}.$$

Since $\delta^{-1} = \max_{1 \leq i \leq N} n_i \geq n_{i'}$, we have $\delta \leq \dfrac{1}{n_{i'}}$, so $B_\delta(x) \subseteq B_{\frac{1}{n_{i'}}} \subseteq \mathcal{U}_{\alpha'_i}$. Thus our δ has the prescribed property.

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