Practice Quiz 2

18.100B R2 Fall 2010

Closed book, no calculators.

YOUR NAME: ______________________________________

This is a 30 minute in-class exam. No notes, books, or calculators are permitted. Point values are indicated for each problem. Do all the work on these pages.

GRADING
1. _______ /15
2. _______ /20
3. _______ /10
4. _______ /20
TOTAL /65

Problem 1. [5+5+5 points]

Let (X, d) be a metric space.

(a) State the definition of a connected subset of X via separated sets, as in Rudin.

(b) Let (X, d) be connected (i.e. X is connected as a subset of (X, d)). Show that a subset $A \subset X$ is both open and closed if and only if $A = \emptyset$ or $A = X$. (This was a homework problem, but the task is to reprove this fact.)

(c) Suppose that (X, d) is a metric space with the following property: A subset $A \subset X$ is both open and closed if and only if $A = \emptyset$ or $A = X$. Then show that (X, d) is connected (i.e. X is connected as a subset of (X, d)).

Problem 2. [10+10 points]

(a) Find $\liminf_{n\to\infty}$ and $\limsup_{n\to\infty}$ for each of the following sequences.

Are these sequences bounded and/or convergent?

$$a_n = \sin\left(\tfrac{n\pi}{4}\right), \qquad b_n = \frac{(-1)^n}{n^{3/2}}.$$

(b) Let (a_n), (b_n) and (c_n) be sequences in $\mathbb{R}$ such that for all $n \geq N$ we have $a_n \leq b_n \leq c_n$. Assume also that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ for some real number L.

Prove that $\lim_{n\to\infty} b_n = L$.

Problem 3. [10 points] Assume that $\sum_{n=1}^{\infty} a_n$ is a convergent series and that $a_n \geq 0$ for all $n \geq N$. Prove that $\sum_{n=1}^{\infty} \frac{1}{n}\sqrt{|a_n|}$ converges. (Hint: You can use the general inequality $2xy \leq x^2 + y^2$ for $x, y \in \mathbb{R}$.)

Problem 4. [20 points: $+4$ for each correct, -4 for each incorrect; no proofs required.] (Hint: Note the penalty – it may be wise to leave some questions unanswered.)

a) Let (X, d) be a metric space, and let $E \subset X$. Then the closure of E is equal to the set $L(E)$ of all limits of sequences in E:

$$L(E) = \{x \in X \mid \exists (x_n)_{n \in \mathbb{N}} \subset E : \lim_{n \to \infty} x_n = x\}.$$

TRUE FALSE

b) If $\sum_{n=1}^{\infty} a_n$ is convergent and $a_n \geq 0$ then $a_n \to 0$.

TRUE FALSE

c) The subset $\{z \in \mathbb{Q} \mid |z| < 1\}$ of $\mathbb{Q}$ is connected.

TRUE FALSE

d) Let (x_n) be a sequence in the metric space (X, d) such that $d(x_n, x_{n+1}) \leq \frac{1}{n}$. Then (x_n) is a Cauchy sequence.

TRUE FALSE

e) Suppose $\sum_{n=1}^{\infty} c_n z^n$ is a power series with convergence radius $R = 2$ and such that it converges for $z = 2$. Then it converges for all other $z \in \mathbb{C}$ with $|z| = 2$.

TRUE FALSE