18.100B : Fall 2010 : Section R2
Homework 5

Due Tuesday, October 12, 1pm

Reading: Tue Oct.5 : completeness, Rudin 3.8-20
Thu Oct.7 : construction of the real numbers, notes

1 . Let H be the set of all human beings (past and present). Let R be the usual family relation on H; that is, for $h, k \in H$ we have $(h, k) \in R$ if and only if h and k have a common ancestor. Show that R is reflexive and symmetric, but not transitive.

2 . (a) Let (a_n) be a Cauchy sequence in $\mathbb{Q}$, such that a_n does *not* converge to 0. Show that there is an $\omega > 0$ and an $N \in \mathbb{N}$ so that $|a_n| > \omega$ for all $n \geq N$. Moreover, show that the sign of a_n is constant for large n.

(b) Following part **(a)**, show that if (a_n) is a Cauchy sequence in $\mathbb{Q}$ that does not converge to 0, then $(1/a_n)$ is a Cauchy sequence in $\mathbb{Q}$. [*Remark*: it would be tempting to solve this problem by noting that, thinking of (a_n) living in the bigger complete metric space $\mathbb{R}$, we can replace the word "Cauchy" by "convergent", which makes for an easy proof. That is *not* the intention of the problem; you should work directly within $\mathbb{Q}$.]

3 . Problem 4, p. 78 in Rudin. (Hint: consider the two subsequences given by the odd and even terms – you can find relations between the $\limsup$ of these two.)

4 . Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Show that the definition of $\liminf$ in the lecture coincides with the s_* of Rudin 3.16. In the lecture, we defined

$$\liminf_{n\to\infty} s_n := \begin{cases} \lim_{n\to\infty} \inf\{s_k \mid k \geq n\} & ;\text{if limit of infima exists,} \\ -\infty & ;\text{if } (s_n) \text{ is not bounded below;} \\ \infty & ;\text{if } s_n \to \infty. \end{cases}$$

5 . Let $\mathscr{P}(\mathbb{N})$ denote the set of all subsets of $\mathbb{N}$. Define a relation $\mathscr{D}$ on $\mathscr{P}(\mathbb{N})$ as follows: $(A, B) \in \mathscr{D}$ if the symmetric difference

$$A\Delta B := (A \cup B) \setminus (A \cap B)$$

is finite. (That is: A and B only differ by finitely many numbers.)

(a) Show that $\mathscr{D}$ is an equivalence relation.

(b) Show that all finite subsets of $\mathbb{N}$ are in the same equivalence class of $\mathscr{D}$.

(c) Show that each equivalence class of $\mathscr{D}$ is a *countable* family of subsets. [*Hint*: fix any subset $A_0 \in \mathscr{P}(\mathbb{N})$, and for each $n \in \mathbb{N}$ consider the classes $A_0^{(n)} = \{A \subseteq \mathbb{N}\,;\, A \text{ differs from } A_0 \text{ by } n \text{ numbers}\}$. Show that $A_0^{(n)}$ is countable, and that the equivalence class $[A_0]_{\mathscr{D}}$ is the union of the countably-many $A_0^{(n)}$.]

not for credit: Conclude that there are uncountably many equivalences classes of $\mathscr{D}$.