18.100B Problem Set 8

Due Thursday November 9, 2006 by 3 PM

Problems:

1) Let $f : [0, \infty) \to \mathbb{R}$ be continuous, and suppose
$$f(x^2) = f(x)$$
holds for every $x \geq 0$. Prove that f has to be a constant function.
Hint: Show that $f(0) = f(x)$ if $0 \leq x < 1$, and $f(1) = f(x)$ if $x \geq 1$.

2) Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x + 1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

3) If
$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$
where $C_0, \ldots, C_n$ are real constants, prove that the equation
$$C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$
has at least one real root between 0 and 1.

4) Suppose f is a real function defined on $\mathbb{R}$. We call $x \in \mathbb{R}$ a *fixed point* of f if $f(x) = x$.
   a) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has *at most* one fixed point.
   b) Show that the function f defined by
   $$f(t) = t + (1 + e^t)^{-1}$$
   has no fixed point, although $0 < f'(t) < 1$ for all real t.
   c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and
   $$x_{n+1} = f(x_n)$$
   for $n = 1, 2, 3, \ldots$

5) Let f be a continuous real function on $\mathbb{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?

6) Let f be a real function on $[a, b]$ and suppose $n \geq 2$ is an integer, $f^{(n-1)}$ is continuous on $[a, b]$, and $f^{(n)}(x)$ exists for all $x \in (a, b)$. Moreover, assume there exists $x_0 \in (a, b)$ such that
$$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0, \quad f^{(n)}(x_0) = A \neq 0.$$
Prove the following criteria: If n is even, then f has a local minimum at x_0 when $A > 0$, and f has a local maximum at x_0 when $A < 0$. If n is odd, then f does not have a local minimum or maximum at x_0. *Hint:* Use Taylor's theorem.

7) For $f(x) = |x|^3$, compute $f'(x)$, $f''(x)$ for all real x, and show that $f^{(3)}(0)$ does not exist.

Extra problems:

1) Let $I \subseteq \mathbb{R}$ be an open interval. A function $f : I \to \mathbb{R}$ is called *Hölder continuous* of order $\alpha > 0$ if there is constant $C > 0$ such that
$$|f(x) - f(y)| \leq C|x - y|^\alpha$$
holds for all $x, y \in I$.
   a) Show that any Hölder continuous function is uniformly continuous.
   b) Prove that $f(x) = \sqrt{|x|}$ defined on $I = \mathbb{R}$ is Hölder continuous of order $\alpha = 1/2$.
   c) Prove that Hölder continuity of order α implies Hölder continuity of order $0 < \beta \leq \alpha$, provided that I is bounded. What happens if I is unbounded?
   d) Show that if f is Hölder continuous of order $\alpha > 1$, then f has to be constant.

2) Let $a \in \mathbb{R}$, and suppose $f : (a, \infty) \to \mathbb{R}$ is a twice-differentiable. Define
$$M_0 = \sup_{a<x<\infty} |f(x)|, \quad M_1 = \sup_{a<x<\infty} |f'(x)|, \quad M_2 = \sup_{a<x<\infty} |f''(x)|,$$
which we assume to be finite numbers. Prove the inequality
$$M_1^2 \leq 4M_0M_2.$$
Hint: If $h > 0$, Taylor's theorem shows that
$$f'(x) = \frac{1}{2h}[f(x + 2h) - f(x)] - hf''(\xi)$$
for some $\xi \in (x, x + 2h)$. Hence
$$|f'(x)| \leq hM_2 + \frac{M_0}{h}.$$
To show that $M_1^2 = 4M_0M_2$ can actually happen, take $a = -1$, define
$$f(x) = \begin{cases} 2x^2 - 1 & \text{if } -1 < x < 0, \\ \dfrac{x^2 - 1}{x^2 + 1} & \text{if } 0 \leq x < \infty. \end{cases}$$
and show that $M_0 = 1$, $M_1 = 4$, $M_2 = 4$.

3) Suppose f is a real, three times differentiable function on $[-1, 1]$, such that
$$f(-1) = 0, \quad f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0.$$
Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$.
Hint: Use Taylor's theorem, to show that there exist $s \in (0, 1)$ and $t \in (-1, 0)$ such that
$$f^{(3)}(s) + f^{(3)}(t) = 6.$$