18.100B : Fall 2010 : Section R2
Homework 6

Due Tuesday, October 19, 1pm

Reading: Tue Oct.12 : series, Rudin 3.20-37
Thu Oct.14 : series, Rudin 3.38-55.

1 . (a) Rudin 6 problem (b) on page 78

(b) Rudin 6 problem (c) on page 78

(c) Prove that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

(Hint: The partial sums can be written as telescoping sum $(a_1 - a_2) + (a_2 - a_3) + ... + (a_{n-1} - a_n) = a_1 - a_n$.)

2 . Assume that $a_n, b_n > 0$ for all $n \geq n_0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$. Prove that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge. (This result is the "limit comparison test".)

3 . (a) Let $N \geq 1$ and let $a_1, a_2, \dots, a_N$ and $b_1, b_2, \dots, b_N$ be real numbers. Verify that

$$\left(\sum_{i=1}^{N} a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^{N} a_i^2\right)\left(\sum_{j=1}^{N} b_j^2\right)$$

and conclude the Cauchy-Schwarz inequality

$$\left|\sum_{i=1}^{N} a_i b_i\right| \leq \left(\sum_{i=1}^{N} a_i^2\right)^{1/2} \left(\sum_{j=1}^{N} b_j^2\right)^{1/2}.$$

Then use the Cauchy-Schwarz inequality inequality to prove the triangle inequality

$$\left(\sum_{i=1}^{N} (a_i + b_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^{N} a_i^2\right)^{1/2} + \left(\sum_{j=1}^{N} b_j^2\right)^{1/2}.$$

(Hint: square both sides.)

(b) Let now

$$X = \left\{ a : \mathbb{N} \to \mathbb{R} \mid \textstyle\sum_{n=1}^{\infty} a(n)^2 \text{ converges} \right\} = \left\{ (a_n)_{n\in\mathbb{N}} \subset \mathbb{R} \mid \textstyle\sum_{n=1}^{\infty} a_n^2 < \infty \right\}$$

and define a norm and induced metric

$$\|(a_n)_{n\in\mathbb{N}}\|_2 = \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2}, \qquad d_2((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) = \|(a_n - b_n)_{n\in\mathbb{N}}\|_2.$$

Use part a) to show that (X, d_2) is a metric space. (Hint: One way to solve this is to first prove that $(X, \|\cdot\|_2)$ is a normed vector space.)

4 . (a) Rudin problem 9(a) and (c) on page 79. (Hint:ratio test)

(b) For both power series, also investigate the convergence on the border of the radius of convergence (for $|z| = R$).

5 . Banach fixed-point theorem:

Let (X, d) be a complete metric space. Suppose $f: X \to X$ has the property that, for some number $c \in (0, 1)$,

$$d(f(x), f(y)) \leq c \cdot d(x, y) \quad \text{for all} \quad x, y \in X.$$

(a) Suppose $y_n \in X$ is any convergent sequence, with limit y. Prove that $f(y_n)$ is a convergent sequence, and $f(y_n) \to f(y)$.

(b) Fix any point $x_0 \in X$, and iteratively define $x_{n+1} = f(x_n)$ for each $n \in \mathbb{N}$. Show that

$$\sum_{j=0}^{\infty} d(x_{j+1}, x_j)$$

is a convergent series. [*Hint*: it is bounded above by a geometric series.]

(c) Show that the sequence (x_n) of iterates of f starting at x_0, as above, is a Cauchy sequence. Conclude that it converges to some point $x_\infty \in X$. [*Hint*: Let $m, n \in \mathbb{N}$, and suppose $m \geq n$. Then $m = n + k$ for some $k \in \mathbb{N}$. Show that $d(x_m, x_n) \leq c^n \cdot d(x_k, x_0)$, and $d(x_k, x_0) \leq d(x_k, x_{k-1}) + d(x_{k-1}, x_{k-2}) + \cdots + d(x_1, x_0) \leq \sum_{j=0}^{\infty} d(x_{j+1}, x_j)$. Conclude that $d(x_m, x_n) \to 0$ as $m, n \to \infty$.]

(d) Show that, given any starting point x_0, the limit x_∞ of the sequence of iterates in **(c)** is a *fixed-point* of f: i.e. $f(x_\infty) = x_\infty$. [*Hint*: using part **(a)**, compare $d(f(x_\infty), x_\infty)$ to $d(f(x_n), x_n)$.]

(e) Let x_0, y_0 be any two points in X. Let $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ be the sequences of iterates. Show that $d(x_n, y_n) \to 0$ as $n \to \infty$. Conclude that $x_\infty = y_\infty$, and that f has a *unique* fixed point.