Recitation 06

To-do list:

1. First we will answer any questions there may be on PSETs and lectures.
2. There are a few remarks about open and closed sets to cover.
3. Then we will discuss lim sup and lim inf.

First we talked about Problem 7B. Given $E \subset \mathbb{R}$, we say that $x \in \mathbb{R}$ is a cluster point of E if $\forall \epsilon$, $(x-\epsilon, x+\epsilon) \cap (E \setminus \{x\}) \neq \emptyset$.

Problem 4

We want to show that the set of cluster points of E (denoted as C) is closed.

To show a set is closed, we need to show that the *complement* of said set is open. What is the complement of the set of cluster points of E? Well, negate the statement of the definition of a cluster point: $x \in \mathbb{R}$ is **not** a cluster point if **there exists an** ϵ such that $(x-\epsilon, x+\epsilon) \cap (E \setminus \{x\}) = \emptyset$.

Remark 5. *Note that when you negate a statement, every $\forall$ changes to an $\exists$. For example, let's say we wanted to show that the statement "for all $x \in \mathbb{R}$ $2x > x$" if a false statement. Then, we simply have to find *one* x that makes this false.*

So we want to find an ϵ' such that $(x-\epsilon', x+\epsilon') \subset C^c$. Can you argue that the x values that have this property are **not** cluster points? This exercise is left to the student.

Now lets discuss some things about open and closed sets. If we want to prove that something is closed, we want to show the complement is open. Hence, to show that a set A is closed, A^c must be open. So, for all $x \notin A$, $\exists \epsilon$ such that

$$(x-\epsilon, x+\epsilon) \subset A^c \iff (x-\epsilon, x+\epsilon) \cap A = \emptyset.$$

Another way to prove that a set A is closed is to show that A contains all of it's limit points. In other words, if given a sequence $\{x_n\}_n \subset A$, and $x_n \to x$, then $x \in A$.

However, note that showing a set is not open **does not** show that the set is closed. For example, the sets $\mathbb{R}$ and $\emptyset$ are both open and closed, and the set $(0,1]$ is neither open nor closed. Keep this in mind when working on questions regarding open and closed sets.

Now lets discuss lim sup and lim inf. We have seen that if a sequence is strictly increasing and bounded, it must converge to the supremum, but not all sequences are increasing. Consider the sequence:

$$\{2, -1, 1+\frac{1}{2}, -\frac{1}{2}, \ldots, 1+\frac{1}{n}, -\frac{-1}{n}\}.$$

We can visually depict this sequence:

It is clear that the infimum is -1, and the supremum is +2, but this doesn't say much about what x_n does as $n \to \infty$. This is why we begin to care about lim inf and lim sup. If we exclude the first $2n$ terms in our sequence, we get that $\inf = -\frac{1}{n}$ and $\sup = 1 + \frac{1}{n}$. Hence, taking $n \to \infty$, we can get that

$$\liminf x_n = 0 \quad \text{and} \quad \limsup x_n = 1.$$

This tells us quite a bit more about the sequence $\{x_n\}_n$.

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