Recitation 04

To-do list:

1. Find the limit of a polynomial (essentially showing that polynomials are continuous).

Let $P(x) = \sum_{i=0}^{n} c_i x^i$ be a polynomial with real coefficients c_i and let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} a_n = a$. We want to show that $\lim_{n\to\infty} P(a_n) = P(a)$. To put this more rigorously (i.e. in real analysis terms), we want to show that for all $\epsilon > 0$, $\exists N$ such that

$$|P(a_n) - P(a)| < \epsilon$$

for all $n > N$.

All we know so far is that $\lim_{n\to\infty} a_n = a$, meaning that for all $\epsilon' > 0$, $\exists N'$ such that $|a_n - a| < \epsilon'$ for all $n > N'$. Note that we use ϵ' and N' so we don't confuse our notation.

To start the problem, now that we have written out everything we are given, we can rewrite "what we want" using the definition of $P(x)$:

$$\sum_{i=0}^{n} c_i a_n^i - \sum_{i=0}^{n} c_i a_n^i \; < \epsilon.$$

This on it's own is quite messy! We can try to simplify it using properties of limits. Recall the following two properties of limits:

1. If $\lim x_n = x$ and $\lim y_n = y$, then $\lim(x_n + y_n) = x + y$.
2. If $\lim x_n = x$ then $\lim(cx_n) = cx$ for all $c \in \mathbb{R}$.

If we could show that $\lim_{n\to\infty} a_n^i = a^i$, then we would get our result. Assume we could show this property of limits– then, by our second property of limits above, $\lim c_i a_n^i = c_i a^i$. Thus (using property 1 here now)

$$\boxed{\lim_{n\to\infty} P(a_n) = \lim_{n\to\infty} \sum_{i=1}^{n} c_i a_n^i = \sum_{i=1}^{n} \lim_{n\to\infty} c_i a_n^i = \sum_{i=1}^{n} c_i a^i = P(a).}$$

So let's actually show that

$$\lim_{n\to\infty} a_n^i = a^i \;\forall i \in \mathbb{N}.$$

In other words, we want to show that for all $\epsilon > 0$, $\exists N$ such that

$$|a_n^i - a^i| < \epsilon \;\forall n > \mathbb{N}.$$

Now note that

$$(a_n^i - a^i) = (a_n - a)(a_n^{i-1} + a_n^{i-2}a + \cdots + a_n a^{i-2} + a^{i-1}).$$

We can intuitively know we want to use this (seemingly random) fact as we want to utilize the fact that $a_n \to a$. Hence,

$$\begin{aligned} |a_n^i - a^i| &= |(a_n - a)(a_n^{i-1} + a_n^{i-2}a + \cdots + a_n a^{i-2} + a^{i-1})| \\ &\leq |a_n - a| \cdot \left(|a_n^{i-1}| + |a_n^{i-2}a| + \cdots + |a_n a^{i-2}| + |a^{i-1}|\right) \end{aligned}$$

by the triangle inequality. Now recall that if a sequence converges, then it is bounded. Hence, there exists a $B \geq 0$

such that $|a_n| \leq B$ for all $n \in \mathbb{N}$. Furthermore,

$$|a| = \lim_{n\to\infty} |a_n| \leq \lim_{n\to\infty} B = B \implies |a|^i \leq B^i.$$

Therefore,

$$|a_n^{i-1}| + |a_n^{i-2}a| + \cdots + |a_n a^{i-2}| + |a^{i-1}| \leq B^{i-1} + B^{i-1} + \cdots + B^{i-1} = i \cdot B^{i-1}$$

where i is a fixed natural number. Hence,

$$|a_n^i - a^i| \leq |a_n - a| \cdot iB^{i-1}$$

where iB^{i-1} is simply a constant. Combining this information with the fact that $a_n \to a$, we have the following: $\forall \epsilon' > 0$, $\exists N'$ such that $|a_n - a| < \epsilon'$ for all $n > N'$. Hence, for all $n \geq N'$

$$\begin{aligned} |a_n^i - a_i| &\leq |a_n - a| \cdot iB^{i-1} \\ &< \epsilon' \cdot iB^{i-1}. \end{aligned}$$

Since this is true for all $\epsilon' > 0$, let $\epsilon' = \frac{\epsilon}{iB^{i-1}}$. Thus,

$$\begin{aligned} |a_n^i - a_i| &\leq |a_n - a| \cdot iB^{i-1} \\ &< \epsilon' \cdot iB^{i-1} \\ &= \epsilon. \end{aligned}$$

We are done! We have shown that $\lim_{n\to\infty} a_n^i = a^i$, and as we have discussed before this finishes our problem.

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